Lebesgue Measure

Introduction

The length $l(I)$ of an interval I is defined to be the difference of the endpoints of the interval

Thus $l(I) = b-a$ where I is any one of $[a,b]$, (a,b), $(a,b]$, $[a,b)$

Thus length is a function i.e.
$l : S \longrightarrow [0,\infty)$, where S is the set of all intervals of $\mathbb{R}$.

s.t.

(i) $l(I) \geq 0 \quad \forall$ intervals I

(ii) If $\{I_i\}$ is a countable collection of mutually disjoint intervals s.t. $\bigcup_i I_i$ is an interval, then

$$l(\bigcup_i I_i) = \sum_i l(I_i)$$

(iii) If x is any fixed real number, then
$$l(I+x) = l(I)$$

Our aim is to extend the notion of length to more complicated sets than intervals for example

Let O be an open set in $\mathbb{R}$. Then O can be written as a countable union of mutually disjoint open intervals $\{I_i\}$ s.t

$$O = \bigcup_i I_i$$

So we can define $l(O) = \sum_i l(I_i)$

But the class of open sets is also restricted.

So we would like to define a fn. m on some collection $\mathcal{M}$ of sets of real numbers with properties of length function.

i.e. we want to define

$$m : \mathcal{M} \rightarrow [0, \infty) \text{ s.t.}$$

(1) mE is defined for each set E of real numbers. i.e. $\mathcal{M} = P(\mathbb{R})$.

(2) for an interval I, $m(I) = l(I)$;

(3) If $\{E_n\}$ is a sequence of disjoint sets then $m(\bigcup E_n) = \sum m(E_n)$

(4) m is translation invariant i.e. if E is the set for which m is defined and if $E + y = \{x + y ; x \in E\}$, then

$$m(E+y) = m(E)$$

We will weaken the property (1) and try to define m for as many sets as possible and will try that this collection is a σ-algebra in $\mathbb{R}$.

σ-algebra:-

A collection $\mathcal{A}$ of subsets of X is called σ-algebra of sets if or Borel field if

(i) $A \cup B \in \mathcal{A}$ if $A, B \in \mathcal{A}$

(ii) $A^c \in \mathcal{A}$ if $A \in \mathcal{A}$

(iii) if $\langle A_i \rangle$ is a sequence in $\mathcal{A}$ then

$$\bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$$

(countable union)

* Outer Measure or Henri Lebesgue Outer Measure.

Def let A be any subset of set $\mathbb{R}$ of real numbers We define $m^*(A)$, the Lebesgue Outer Measure or Simply Outer measure of A as follows:

$$m^*(A) = \inf \left\{ \sum_{n=1}^{\infty} |I_n| \right\}$$

where infimum is taken w.r.t all Countable Collections $\{I_n\}$ of open intervals such that

$$A \subseteq \bigcup_{n=1}^{\infty} I_n$$

i.e $\{I_n\}$ is a covering of A and $|I_n|$ denotes $l(I_n$

Property-I $m^* \phi = 0$.

Proof. let $\epsilon > 0$ be any number. Then

$$\phi \subseteq (-\epsilon/2, \tfrac{\epsilon}{2})$$

$$\Rightarrow m^*(\phi) \le l\left(-\tfrac{\epsilon}{2}, \tfrac{\epsilon}{2}\right)$$

$$\Rightarrow m^*(\phi) \le \epsilon$$

but $\epsilon > 0$ is arbitrary.

~~also~~ $\Rightarrow m^*(\phi) \le 0$

also $m^*(\phi) \ge 0$

Hence $m^*(\phi) = 0$.

Property-II If A and B are two sets with $A \subset B$ then $m^*(A) \le m^*(B)$.

Proof let $\{I_n\}$ be a countable collection of disjoint

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open intervals such that

$$B \subset \bigcup_n I_n$$

Also $A \subset B \Rightarrow A \subset \bigcup_n I_n$

Taking outer measure on both sides, we get

$$m^*(A) \le \sum_n \ell(I_n)$$

$$\Rightarrow m^*(A) \le \inf.\left(\sum_n \ell(I_n)\right)$$

$$= m^*(B)$$

Hence proved.

Property-3 Outer measure of every singleton set is ze

Proof. let $A = \{x\}$

let $\epsilon > 0$ be given.

then if we take $I = (x - \epsilon/2, x + \epsilon/2)$, then

$A \subset I$

$\Rightarrow m^*(A) \le \ell(I) = \epsilon$

but $\epsilon > 0$ is arbitrary small no.

so $m^*(A) \le 0$

also $m^*(A) \geq 0$

Hence $m^*(A) = 0$.

Proposition The outer measure of an interval is its length.

Proof. Case 1 Suppose I is finite closed interval

i.e.

$$I = [a, b]$$

let $\epsilon > 0$ be given. Then

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Heine - Borel Thm. Every closed and bdd subset of $\mathbb{R}$ is compact.

$I = [a,b] \subseteq (a-\epsilon/2, b+\epsilon/2)$

$\Rightarrow m^*(I) \leq l(a-\epsilon/2, b+\epsilon/2)$

$\Rightarrow m^*(I) \leq b-a+\epsilon$

but ϵ is arbitrary small no. so
$m^*(I) \leq b-a = l(I)$

now we will show that $m^*(I) \geq b-a$
For this we show that if $\{I_n\}$ is any countable collection of open intervals covering $[a,b]$, then

$$\sum l(I_n) \geq b-a \qquad \text{---------} \quad (1)$$

By Heine - Borel thm., as $[a,b]$ is closed & bdd, it must be compact and so any collection of open intervals covering $[a,b]$ contains a finite subcollection which also covers $[a,b]$
Also sum of lengths of finite subcollection of intervals $\leq$ sum of lengths of countable covering
So we need to prove (1) only for finite collections $\{I_n\}$ which cover $[a,b]$.

Now $a \in [a,b] \subseteq \cup I_n$
$\Rightarrow a \in I_n$ for some n
let $a \in (a_1, b_1)$
$\Rightarrow a_1 < a < b_1$
If $b_1 \leq b$ then $b_1 \in [a,b]$
and also $b_1 \notin (a_1, b_1)$
$\Rightarrow \exists\ (a_2, b_2)$ in collection $\{I_n\}$ s.t.
$b_1 \in (a_2, b_2) \Rightarrow a_2 < b_1 < b_2$

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Continuing this way, we obtain a sequence $(a_1, b_1), (a_2, b_2), \ldots (a_k, b_k)$ from the collection $\{I_n\}$ s.t.

$$a_i < b_{i-1} < b_i \quad , i = 1, 2, \ldots \qquad \text{and } b_0 = a$$

Since $\{I_n\}$ is a finite collection, this process must terminate with some interval (a_k, b_k). and it terminates only if $b \in (a_k, b_k)$ i.e. $a_k < b < b_k$. Thus

$$\sum_n l(I_n) \geq \sum_{i=1}^{k} l(a_i, b_i)$$

$$= (b_k - a_k) + (b_{k-1} - a_{k-1}) + \cdots + (b_2 - a_2) + (b_1$$

$$= b_k - (a_k - b_{k-1}) - (a_{k-1} - b_{k-2}) - \cdots - (a_2 - b_1$$

$$> b_k - a_1 \qquad [\because a_i < b_{i-1}]$$

but $b_k > b$ and $a > a_1$

$$\Rightarrow b_k - a_1 > b - a.$$

Hence $\sum l(I_n) > b - a$

Hence (1) is proved and $m^*[a, b] = b - a$.

2 Suppose I is any finite interval Then given $\epsilon > 0$, $\exists$ a closed interval $J \subset I$ s.t.

$$l(J) > l(I) - \epsilon \qquad [\because \text{eg. if } J = (a,b) \text{ take } J = [a + \tfrac{\epsilon}{\cdot}$$

$$\Rightarrow l(I) - \epsilon < l(J) = m^*(J) \leq m^* I \leq m^* \bar{I}$$

$$= l(\bar{I}) = l(I)$$

Hence $l(I) - \epsilon < m^*(I) \le l(I)$
and $\epsilon > 0$ is arbitrary
so $m^*(I) = l(I)$

Case 3 If I is an infinite interval.
Then given any real number $k > 0$, $\exists$ a closed finite interval $J \subset I$ s.t. $l(J) = k$.
Hence

$$m^*(I) \geq m^*(J) = l(J) = k$$

$\Rightarrow m^*(I) \geq k$ for each positive real no k

$$\Rightarrow m^*(I) = \infty = l(I)$$

Hence proved.

Proposition-2 Let $\{A_n\}$ be a countable collection of sets of real numbers Then

$$m^*(\cup A_n) \leq \sum m^*(A_n)$$

Proof If $m^*(A_n) = \infty$ for some $n \in \mathbb{N}$, then the inequality holds trivially.

So assume that $m^*(A_n) < \infty$ for each $n \in \mathbb{N}$.
Then for each n and for a given $\epsilon > 0$, $\exists$ a countable collection $\{I_{n,i}\}_i$ of open intervals such that $A_n \subset \bigcup_i I_{n,i}$ and

$$m^* A_n + 2^{-n}\epsilon > \sum_i l(I_{n,i}) \quad —(1)$$

[$\because m^*(A_n)$ is $\inf(\sum I_n)$ but $m^*(A_n) + 2^{-n}\epsilon$ is not inf. so at least one element is smaller]

also as $A_n \subset \bigcup_i I_{n,i}$

$$\Rightarrow \bigcup_n A_n \subset \bigcup_n \bigcup_i I_{n,i}$$

and collection $\{I_{n,i}\}_{n,i}$ is a countable collec-
of open intervals.

($\because$ countable union of countable sets is countable).

Hence $m^*(\bigcup_n A_n) \le \sum_n \sum_i l(I_{n,i})$

$$\Rightarrow m^*(\bigcup_n A_n) < \sum_n (m^* A_n + \epsilon 2^{-n})$$ (fro

$$= \sum_n m^* A_n + \epsilon \sum_n 2^{-n} \quad —(2)$$

also $\sum_n 2^{-n} = \frac{1}{2} + \frac{1}{2^2} + \ldots = \frac{\frac{1}{2}}{1-\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$

So from (2)

$$m^*(\bigcup_n A_n) < \sum_n m^* A_n + \epsilon$$

but ϵ was arbitrary +ve number

$$\Rightarrow m^*(\bigcup_n A_n) \le \sum_n m^*(A_n).$$

<u>Corollary</u> - If A is countable, $m^*(A) = 0$

<u>Proof</u> Let $A = \{a_1, a_2, \ldots\}$ be the countable set

Then $A = \bigcup_i A_i$ where $A_i = \{a_i\}$, $i = 1, 2, \ldots$

then by previous result

$$m^*(A) = m^*(\bigcup_i A_i) \le \sum_i m^*(A_i)$$

$\Rightarrow m^*(A) \leq 0+0+0+\cdots+0$ [$\because m^*(\text{singleton}) = 0$]

$\Rightarrow m^*(A) \leq 0$

also $m^*(A) \geq 0$

Hence $m^*(A) = 0$.

Independent Proof

Proof let $A = \{a_1, a_2, \ldots\}$

Consider $I_i = \left(a_i - \frac{\epsilon}{2^{i+1}}, a_i + \frac{\epsilon}{2^{i+1}}\right) \; \forall \; i = 1, 2, \ldots$

then $A \subset \bigcup_i I_i$

$$\Rightarrow m^*(A) \leq \sum_{i=1}^{\infty} l\left(a_i - \frac{\epsilon}{2^{i+1}}, a_i + \frac{\epsilon}{2^{i+1}}\right)$$

$$= \sum_{i=1}^{\infty} \frac{\epsilon}{2^i} = \epsilon \sum_{i=1}^{\infty} \frac{1}{2^i}$$

$$= \epsilon$$

$\Rightarrow m^*(A) \leq \epsilon$

but $\epsilon > 0$ is arbitrary small no.

$\Rightarrow m^*(A) \leq 0$

but $m^*(A) \geq 0$ is always true

Hence $m^*(A) = 0$.

Corollary The set $[0,1]$ is not countable

Proof If $[0,1]$ is countable then $m^*[0,1] = 0$

but $m^*[0,1] = l[0,1] = 1 - 0 = 1 \neq 0$

Hence $[0,1]$ is not countable.

CONVERSE NEED NOT BE TRUE

i.e.
If outer measure of a set is zero, then the set need not be countable.

eg. Cantor Set.

To construct Cantor set, consider $[0,1]$

then $[0,1] = \left[0, \frac{1}{3}\right] \cup \left(\frac{1}{3}, \frac{2}{3}\right) \cup \left[\frac{2}{3}, 1\right]$

(i.e. divide $[0,1]$ in three parts)

let $C_1 = \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$

$= \left[0, \frac{1}{9}\right] \cup \left(\frac{1}{9}, \frac{2}{9}\right) \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left(\frac{7}{9}, \frac{8}{9}\right)$
$\cup \left[\frac{8}{9}, 1\right]$

$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$

similarly construct $C_3, C_4, C_5, \ldots$

then cantor set $C = \bigcap_{n=1}^{\infty} C_n$

Note that C_1 is union of 2 sets each of length $\frac{1}{3}$
C_2 " " " 2^2 sets " " " $\frac{1}{3^2}$
⋮
C_n " " " 2^n sets " " " $\frac{1}{3^n}$

$m^*(C) = m^*\left(\bigcap_{n=1}^{\infty} C_n\right) \leq m^*(C_n)$ $\left[\because \bigcap C_n \subseteq C_n\right]$

$$\Rightarrow m^*(C) \le m^*(C_n) \le \frac{1}{3^n} + \frac{1}{3^n} + \cdots + \frac{1}{3^n} \quad (2^n \text{ times})$$

$$\Rightarrow m^*(C) \le \left(\frac{2}{3}\right)^n \quad \forall n$$

taking $n \to \infty$ $\left[\because \frac{2}{3} < 1 \text{ and } \lim_{n\to\infty} x^n = 0 \text{ if } x<1\right]$

$m^*(C) \le 0$

also $m^*(C) \geq 0$

Hence $m^*(C) = 0$

Hence Outer measure of Cantor set is zero

We will prove that cantor set is uncountable for this, we need ternary representation of a number.

let $m \in \mathbb{N}$

If $m = 3^n a_n + 3^{n-1} a_{n-1} + \cdots + 3^2 a_2 + 3a_1 + 3^0 a_0$

then $m = a_n a_{n-1} \ldots a_1 a_0$ is its ternary representation.

In case of fractions $\frac{a}{b}$ ~~eg. $\frac{1}{3}$~~

write $\frac{a}{b} = C_1 \frac{1}{3} + C_2 \frac{1}{3^2} + C_3 \frac{1}{3^3} + \cdots$

then $\frac{a}{b} = \cdot C_1 C_2 C_3 \ldots$ is its ternary representation

Now we find ternary representation of the end points of open intervals which were left out while constructing cantor set.

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$\frac{1}{3} = \frac{1}{3} \times 1 \Rightarrow$ TR of $\frac{1}{3} = 0.1$

$\frac{2}{3} = \frac{1}{3} \times 2 \Rightarrow$ TR of $\frac{2}{3} = 0.2$

TR ~~of~~ $\frac{1}{9} = \frac{1}{3} \times 0 + \frac{1}{3^2} \times 1 \Rightarrow$ TR of $\frac{1}{9} = .01$

$\frac{2}{9} = \frac{1}{3} \times 0 + \frac{1}{3^2} \times 2 \Rightarrow$ TR of $\frac{2}{9} = 0.02$

$\frac{7}{9} = \frac{1}{3} \times 2 + \frac{1}{3^2} \times 1 \Rightarrow$ TR of $\frac{7}{9} = .21$

$\frac{8}{9} = \frac{1}{3} \times 2 + \frac{1}{3^2} \times 2 \Rightarrow$ TR of $\frac{8}{9} = .22$

and so on.

Note that $C_1 = [0,1] \setminus (0.1, 0.2) = [0,1] \setminus (0.0222\ldots, 0.2)$

$C_2 = C_1 \setminus (0.01, 0.02) \cup (0.21, 0.22)$
$= C_1 \setminus (0.00222\ldots, 0.02) \cup (0.20222\ldots, 0.22)$

so we see that C can have only those points which do not have 1 anywhere in its ternary representation.

[$\because$ any no. can have at most two ternary representations of type $0.d_1 d_2 \ldots$

eg. ~~Only~~ $0.1 = 0.0222\ldots \in C$

but $0.11 = 0.1022\ldots \notin C$]

so $x \in C$
~~then~~ iff $x = .t_1 t_2 \ldots$
where $t_i \in \{0, 2\}$

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[Except $x = \frac{m}{3^n}$, $0 < x < 1$, every real no. in $[0,1]$ has unique ternary representation]

Now define $f: C \to [0,1]$ by

$$f(\cdot t_1 t_2 \ldots) = (\cdot x_1 x_2 \ldots) \xrightarrow{} \text{binary rep}$$

where $x_i = \begin{cases} t_i & \text{if } t_i = 0 \\ \frac{t_i}{2} & \text{if } t_i = 2 \end{cases}$

then f is ^not^ one-one and onto — as $f\left(\frac{1}{3}\right) = 0.0111\ldots = 0.1 = \frac{1}{2} = f\left(\frac{2}{3}\right)$.

As $[0,1]$ is uncountable, so is C.

Borel set: ~~B~~ The collection $\mathcal{B}$ of Borel sets is the smallest σ-algebra which contains all of the open sets.

F_σ-set A set which is a countable union of closed sets is called an F_σ (F for closed, σ for sum) set.

(1) Thus every countable set is an F_σ set.

~~(2)~~ ~~Ev~~ $\because$ countable set = Countable union of singleton sets.

(2) Every closed set is F_σ

(3) Countable union of sets in F_σ is in F_σ.

(4) As $(a,b) = \bigcup_{n=1}^{\infty} \left[a+\frac{1}{n}, b-\frac{1}{n}\right] \in F_\sigma$

$\Rightarrow$ each open interval is an F_σ and as every open set is countable union of disjoint open intervals

$\Rightarrow$ every open set is an F_σ.

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G_δ-set - A set is a G_δ set if it is the intersection of a countable collection of open sets (G for open, δ for durchschnitt).

Hence $(F_\sigma)^c = G_\delta$ and $G_\delta^{\ c} = F_\sigma$

F_σ and G_δ are types of BOREL SETS.

Proposition Given any set A and any $\epsilon > 0$, there is an open set O s.t. $A \subset O$ and $m^* O \leq m^* A + \epsilon$
There is a $G \in G_\delta$ s.t. $A \subset G$ and
$$m^*(A) = m^*(G)$$

Proof (a) ~~Firstly suppose $m^*(A) < \infty$ and~~
let $\epsilon > 0$ be given
Then $\exists$ a countable collection $\{I_n\}$ of open intervals s.t.
$$A \subset \bigcup_n I_n \quad \text{and}$$
$$\sum_n \ell(I_n) \leq m^*(A) + \epsilon \quad -(1)$$

let $O = \bigcup_n I_n$ then O is clearly open.

and $m^*(O) = m^*\left(\bigcup_n I_n\right)$

$\Rightarrow m^*(O) \leq \sum_n \ell(I_n)$ (by countable subadditive property)

So from (1)
$$m^*(O) \leq m^*(A) + \epsilon$$

(b) Choose $\epsilon = \frac{1}{n}$ in (a), where $n \in \mathbb{N}$

Then for each $n \in \mathbb{N}$, $\exists$ an open set $O_n \supset A$ s.t

$$m^*(O_n) \le m^*(A) + \frac{1}{n}$$

let $G = \bigcap_{n=1}^{\infty} O_n$

Then clearly G is a G_δ - set and $G \supset A$

Also

$$m^*(A) \le m^*(G) \qquad [\because A \subset G]$$

$$\le \cancel{m^*()} \; m^*(O_n) \qquad [\because G = \bigcap_{n=1}^{\infty} O_n \subseteq O_n]$$

$$\le m^*(A) + \frac{1}{n} \qquad \forall \, n \in \mathbb{N}$$

Taking $n \to \infty$, we get

$$m^*(G) = m^*(A)$$ Hence proved.

EXERCISES

1. Prove that m^* is translation invariant.
i.e. $m^*(A+x) = m^*(A)$ for every set A and every real number x.

Proof: Let $\epsilon > 0$ be given

Then there is a countable collection $\{I_n\}$ of open intervals s.t. $A \subset \bigcup_n I_n$ and

$$\sum_n l(I_n) \le m^*(A) + \epsilon$$

also as $A \subset \bigcup_n I_n$

$$\Rightarrow A + x \subset \bigcup_n (I_n + x)$$

$$\Rightarrow m^*(A+x) \le \sum_n l(I_n + x) = \sum_n l(I_n) \le m^*(A) + \epsilon$$

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Hence $m^*(A+x) \le m^*(A)+\epsilon$

but $\epsilon > 0$ is arbitrary small

$$\Rightarrow m^*(A+x) \le m^*(A) \quad -(1)$$

similarly If $B = A+x$, then

$$m^*(B - x) \le m^*(B)$$

$$\Rightarrow m^*(A) \le m^*(A+x) \quad -(2)$$

~~He~~ from (1) and (2)

$$m^*(A) = m^*(A+x)$$

~~2. If A and B are any two disjoint subsets of $\mathbb{R}$, then $m^*(A \cup B) = m^*(A) + m^*(B)$~~

~~Proof. By Countable subadditive property of Outer measure,~~

~~$$m^*(A \cup B) \le m^*(A) + m^*(B)$$~~

~~So we only need to prove that $m^*(A \cup B) \ge m^*(A) + m^*(B)$~~

2 If $m^*(A)=0$, then $m^*(A \cup B) = m^*(B)$. In particular if $B \subset A$, then $m^*(B)=0$

Proof.

$$m^*(A \cup B) \le m^*(A) + m^*(B)$$

but $m^*(A)=0$

$$\Rightarrow m^*(A \cup B) \le m^*(B) \quad -(1)$$

Also $B \subseteq A \cup B$

$$\Rightarrow m^*(B) \le m^*(A \cup B) \quad -(2)$$

from (1) and (2), we get

$$m^*(A \cup B) = m^*(B)$$

In particular if $B \subset A \Rightarrow m^*(A \cup B) = m^*(A) = 0 = m^*(B)$

Measurable Sets and Lebesgue Measure

Outer measure has the advantage that is defined for all subsets of real numbers but it is not countably additive.

To make it countably additive, we reduce the family of sets on which it is defined. So we have the following definition due to Caratheodory:

Def. A set E is said to be Lebesgue measurable or measurable if for each set $A \subseteq \mathbb{R}$, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$$

as $m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$ always

$$\left[\because \; A = (A \cap E) \cup (A \cap E^c)\right]$$

So a set E is measurable iff for each $A \subseteq \mathbb{R}$,

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$$

Remark by Def., E is measurable iff E^c is.

1. Prove that ϕ and $\mathbb{R}$ are measurable.

Proof. as for each $A \subseteq \mathbb{R}$

$A = A \cap \phi^c$ and $A \cap \phi = \phi$

$\Rightarrow m^*(A) = m^*(A \cap \phi^c)$ and $m^*(A \cap \phi) = 0$

so $m^*(A) = m^*(A \cap \phi) + m^*(A \cap \phi^c)$

$\Rightarrow \phi$ is measurable.

Similarly $A \cap \mathbb{R} = A$ and $A \cap \mathbb{R}^c = \phi$

$\Rightarrow m^*(A \cap \mathbb{R}) = m^*(A)$ and $m^*(A \cap \mathbb{R}^c) = 0$

so $m^*(A) = m^*(A \cap \mathbb{R}) + m^*(A \cap \mathbb{R}^c)$

Hence $\mathbb{R}$ is measurable.

Lemma 6 If $m^*E = 0$, then E is measurable

Proof for any $A \subseteq \mathbb{R}$

$$A \cap E \subseteq E \Rightarrow m^*(A \cap E) \leq m^*(E) = 0$$

$$\Rightarrow m^*(A \cap E) = 0 \quad \text{--- (1)}$$

and $A \cap E^c \subseteq A$

$$\Rightarrow m^*(A \cap E^c) \leq m^*(A) \quad \text{--- (2)}$$

from (1) and (2)

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c)$$

also $m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$ always holds

$\Rightarrow$ E is measurable.

Lemma 7. If E_1 and E_2 are measurable, so is $E_1 \cup E_2$

Proof as E_2 is measurable $\Rightarrow$ for any $A \subseteq \mathbb{R}$,

$$m^*(A \cap E_1^c) = m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c)$$

$$= m^*\left(A \cap (E_1^c \cap E_2)\right) + m^*\left(A \cap (E_1 \cup E_2)^c\right)$$

Now $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2 \cap E_1^c)$

$$\Rightarrow m^*\left(A \cap (E_1 \cup E_2)\right) \leq m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c)$$

Hence $m^*\left(A \cap (E_1 \cup E_2)\right) + m^*\left(A \cap (E_1 \cup E_2)^c\right)$

$$\leq m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c) + m^*\left(A \cap (E_1 \cup E_2)^c\right)$$

$$= m^*(A \cap E_1) + m^*(A \cap E_1^c)$$

$$= m^*(A)$$

Hence $E_1 \cup E_2$ is measurable.

* Independent Proof that $E_1, E_2 \in \mathcal{M} \Rightarrow E_1 \cap E_2 \in \mathcal{M}$ on Page: 2.

<u>Corollary</u> The family $\mathcal{M}$ of measurable sets is an algebra of sets.

Proof Obvious

($\because$ by def. A collection $\mathcal{A}$ of subsets of X is called an algebra of sets or a Boolean Algebra if

(1) $A \cup B \in \mathcal{A}$ if $A, B \in \mathcal{A}$

(2) $A^c \in \mathcal{A}$ if $A \in \mathcal{A}$

(3) $A \cap B \in \mathcal{A}$ if $A, B \in \mathcal{A}$ (by De-Morgan's law))

<u>Lemma 9</u> let A be any set and $E_1, E_2, \ldots, E_n$ a finite sequence of disjoint measurable sets. Then

$$m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) = \sum_{i=1}^{n} m^*(A \cap E_i)$$

<u>Proof</u> We prove the Lemma by induction on n. clearly the result is true for $n=1$. let's assume that it is true for $n-1$ sets

i.e. $$m^*\left(A \cap \left(\bigcup_{i=1}^{n-1} E_i\right)\right) = \sum_{i=1}^{n-1} m^*(A \cap E_i)$$

Now as E_n is measurable

$$\Rightarrow m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) = m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right) \cap E_n\right) + m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right) \cap E_n^c\right)$$

$$\Rightarrow m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) = m^*(A \cap E_n) + m^*\left(A \cap \bigcup_{i=1}^{n-1} E_i\right)$$

$$= m^*(A \cap E_n) + \sum_{i=1}^{n-1} m^*(A \cap E_i)$$

$$= \sum^{n} m^*(A \cap E_i)$$

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Hence the lemma is true for all n subsets E_i

~~lemma~~

The following proposition will be used to prove next theorem.

<u>Proposition</u> Let $\mathcal{A}$ be an algebra of sets and $\langle A_i \rangle$ a sequence of sets in $\mathcal{A}$. Then there is a sequence $\langle B_i \rangle$ of sets in $\mathcal{A}$ s.t. $B_n \cap B_m = \phi$ for $n \neq m$ and $\bigcup_{i=1}^{\infty} B_i = \bigcup_{i=1}^{\infty} A_i$ (Prop 1.2, Page 18 H. L. Royden)

(Choose $B_n = A_n \sim (A_1 \cup A_2 \cup \cdots \cup A_{n-1})$) (PROOF ON PAGE NE

<u>Theorem</u> The collection $\mathcal{M}$ of measurable sets is a σ-algebra; i.e. the complement of a measurable set is measurable and the union (and $\cap$) of a countable collection of measurable sets is measurable

<u>Proof</u>. We ~~onl~~ have already proved that $\mathcal{M}$ is an algebra of sets.

So we only need to prove that union of countable collection of measurable sets is measurable.

Let E be union of countable collection of measurable sets. Then by above proposition, E is union of sequence $\langle E_n \rangle$ of pairwise disjoint measurable sets.

Let $A \subseteq \mathbb{R}$ and $F_n = \bigcup_{i=1}^{n} E_i \subset E$

Then F_n is measurable and $F_n^c \supset E^c$

$$\Rightarrow m^*(A) = m^*(A \cap F_n) + m^*(A \cap F_n^c)$$

$$\geq m^*(A \cap F_n) + m^*(A \cap E^c) \quad [\because E^c \subset F_n^c$$

$\Rightarrow m^*(A) \geq m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) + m^*(A \cap E^c)$

$= \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap E^c)$ (By previous Lemma)

But LHS is independent of n.

$\Rightarrow m^*(A) \geq \sum_{i=1}^{\infty} m^*(A \cap E_i) + m^*(A \cap E^c)$

$\geq m^*(A \cap E) + m^*(A \cap E^c)$

($\because m^*$ is countable subadditive)

Also $m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c)$

$\Rightarrow m^*(A) = m^*(A \cap E) + m^*(A \cap E^c)$

Hence E is measurable.

Lemma The interval (a, ∞) is measurable.

Proof: let A be any subset of $\mathbb{R}$.

Consider $A \cap (a, \infty) = A_1$ and $A \cap (-\infty, a] = A_2$.

TP $m^*(A) \geq m^*(A_1) + m^*(A_2)$

If $m^*(A) = \infty$, then nothing to prove.

let $m^* A < \infty$

Then given $\epsilon > 0$, $\exists$ a countable collection $\{I_n\}$ of open intervals which covers A and

$m^* A + \epsilon \geq \sum l(I_n)$

Now let $I_n \cap (a, \infty) = I_n'$

and $I_n \cap (-\infty, a] = I_n''$

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Then I_n' and I_n'' are intervals (may be em[pty])
and

$$l(I_n) = l(I_n') + l(I_n'')$$
$$= m^* I_n' + m^* I_n''$$

as $A_1 = A \cap (a,\infty) \subseteq (\cup I_n) \cap (a,\infty)$
$$= \cup\left(I_n \cap (a,\infty)\right) = \cup I_n'$$

$$\Rightarrow m^* A_1 \leq m^*(\cup I_n') \leq \sum m^* I_n'$$

similarly $m^* A_2 \leq \sum m^* I_n''$

$$\Rightarrow m^* A_1 + m^* A_2 \leq \sum (m^* I_n' + m^* I_n'')$$
$$= \sum l(I_n) \leq m^* A + \epsilon$$

but ϵ was arbitrary +ve no.

$$\Rightarrow m^* A_1 + m^* A_2 \leq m^* A$$

Hence proved.

Def. A Borel set is a set which can be formed from open sets (or equivalently from closed sets) through the operations of countable union, countab[le] intersection and relative complement.

Theorem Every Borel set is measurable. In particular each open set and each closed set is measurable.

Proof. As $(a,\infty) \in \mathcal{M}$
and $\mathcal{M}$ is σ algebra

$\Rightarrow \quad (a,\infty)^c = [-\infty, a] \in \mathcal{M}$

also $\quad (-\infty, b) = \bigcup_{n=1}^{\infty} \left(-\infty, b-\frac{1}{n}\right]$

$\Rightarrow (-\infty, b) =$ countable union of sets in σ-algebra $\mathcal{M}$

$\Rightarrow (-\infty, b) \in \mathcal{M}$

Hence $\quad (a,b) = (-\infty, b) \cap (a,\infty) \in \mathcal{M}$.

If A is open set, then A is countable union of open intervals so $A \in \mathcal{M}$.

If B is closed set then B^c is open
$\Rightarrow B^c \in \mathcal{M} \Rightarrow (B^c)^c \in \mathcal{M}$ i.e. $B \in \mathcal{M}$.

Hence σ-algebra $\mathcal{M}$ contains all open sets and so it contains family $\mathcal{B}$ of Borel sets. Hence each Borel set is measurable

LEBESGUE MEASURE If $E \in \mathcal{M}$ then lebesgue Measure (mE) is
$$mE = m^*E$$

Hence $\quad m = m^*|_{\mathcal{M}}$

i.e. m is the set function obtained by restricting the set fn. m^* to the family $\mathcal{M}$ of measurable sets.

There are two important properties of lebesgue Measure:

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Proposition 13 Let $\langle E_i \rangle$ be a sequence of measurable sets. Then $m(\cup E_i) \le \sum m E_i$.

If sets E_n are pairwise disjoint, then

$$m(\cup E_i) = \sum m E_i.$$

Proof. As for a measurable set E, $mE = m^*E$, so inequality is just a restatement of the subadditivity of m^*.
Now if $\{E_i\}$ is a finite sequence of disjoint measurable sets, then by Lemma 9, (taking $A = \mathbb{R}$) we get

$$m(\cup E_i) = \sum m E_i$$

Now let $\{E_i\}$ be infinite sequence of pairwise disjoint measurable sets. Then.

$$\bigcup_{i=1}^{n} E_i \subsetneq \bigcup_{i=1}^{\infty} E_i$$

$$\Rightarrow m\left(\bigcup_{i=1}^{n} E_i\right) \le m\left(\bigcup_{i=1}^{\infty} E_i\right)$$

$$\quad \| $$

$$\sum_{i=1}^{n} m(E_i)$$

but RHS is independent of n

$$\Rightarrow \sum_{i=1}^{\infty} m(E_i) \le m\left(\bigcup_{i=1}^{\infty} E_i\right)$$

Also $m\left(\bigcup_{i=1}^{\infty} E_i\right) \le \sum_{i=1}^{\infty} m(E_i)$ (By Countable subadditivity of m^*)

Hence $m\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} m(E_i)$

Proposition-14 let $\langle E_n \rangle$ be an infinite decreasing sequence of measurable sets, i.e., a sequence with $E_{n+1} \subset E_n$ for each n. Let mE_1 be finite. Then

$$m\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} mE_n$$

Proof let $E = \bigcap_{i=1}^{\infty} E_i$ and let

$F_i = E_i \setminus E_{i+1}$. Then

$$\cancel{E_1 \setminus E = E_1 \cap E^c = E_1 \cap \left(\bigcap_{i=1}^{\infty} E_i\right)^c}$$

$$\bigcup_{i=1}^{\infty} F_i = \bigcup_{i=1}^{\infty} (E_i \setminus E_{i+1}) = \bigcup_{i=1}^{\infty} (E_i \cap E_{i+1}^c)$$

$$= \bigcup_{i=1}^{\infty} E_i \cap \left(\bigcup_{i=1}^{\infty} E_{i+1}\right)^c$$

$$= E_1 \cap \left(\bigcap_{i=1}^{\infty} E_{i+1}\right)^c = E_1 \cap E^c = E_1 \setminus E$$

and F_i's are pairwise disjoint.

$$\Rightarrow m\left(\bigcup_{i=1}^{\infty} F_i\right) = m(E_1 \setminus E)$$

$$\Rightarrow m(E_1 \setminus E) = \sum_{i=1}^{\infty} m(F_i) = \sum_{i=1}^{\infty} m(E_i \setminus E_{i+1}) \quad -(1)$$

but $E_1 = E \cup (E_1 \setminus E)$

$$\Rightarrow mE_1 = mE + m(E_1 \setminus E)$$

similarly $mE_i = mE_{i+1} + m(E_i \setminus E_{i+1})$

so $m(E_1 \setminus E) = mE_1 - mE \quad (\because E \subset E_1)$

and $m(E_i \setminus E_{i+1}) = mE_i - mE_{i+1} \quad (\because mE_i \le mE_1 < \infty)$

Hence from (i)

$$mE_1 - mE = \sum_{i=1}^{\infty} (mE_i - mE_{i+1})$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} (mE_i - mE_{i+1})$$

$$= \lim_{n\to\infty} (mE_1 - mE_n) = mE_1 - \lim_{n\to\infty} mE_n$$

as $mE_1 < \infty$, we get

$$mE = \lim_{n\to\infty} mE_n .$$

Note Result is not true if $mE_1 < \infty$ is removed.

Question ~~Show that by an example, that the condition $m(E_1) < \infty$ can not be dropped in this theorem~~

eg. $E_1 = (0, \infty)$, $E_2 = (2, \infty)$, ---, $E_n = (n, \infty)$. then $E = \cap E_n = \phi \Rightarrow mE = 0 \neq \infty$

Proposition 15 let E be a given set. Then the following five statements are equivalent:

(i) E is measurable

(ii) Given $\epsilon > 0$, $\exists$ an open set $O \supset E$ s.t. $m^*(O \setminus E) < \epsilon$

(iii) Given $\epsilon > 0$, $\exists$ a closed set $F \subset E$ s.t. $m^*(E \setminus F) < \epsilon$

(iv) There is a G in G_δ with $E \subset G$, $m^*(G \setminus E) = 0$

(v) There is an F in F_σ with $F \subset E$, $m^*(E \setminus F) = 0$

If $m^* E$ is finite, the above statements are equivalent to:

(vi) Given $\epsilon > 0$, there is a finite union U of open intervals such that ~~m^*~~ $m^*(U \Delta E) < \epsilon$

Proof We will show that

(i) $\Rightarrow$ (ii) $\Rightarrow$ (iv) $\Rightarrow$ (i)

(i) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (i)

and then (i) $\Rightarrow$ (vi) and ~~(vi) $\Rightarrow$ (i)~~

(i) $\Rightarrow$ (ii)

<u>Case I</u> Let $m^*E < \infty$.

Then given $\epsilon > 0$, $\exists$ Countable collection $\{I_n\}$ of open intervals s.t. $E \subset \bigcup_n I_n$ and $\sum \ell(I_n) < m^*E + \epsilon$

Let $O = \bigcup I_n$ then O is open and $O \supset E$.

$m^*(O) \leq \sum_n m^*(I_n) = \sum_n \ell(I_n) < m^*E + \epsilon$.

$\Rightarrow m^*(O) < m^*E + \epsilon$ —— (1)

but as $O \supset E$ and $m^*E < \infty$

$\Rightarrow m^*(O) < \infty$

$\Rightarrow m^*(O \setminus E) = m^*(O) - m^*E < \epsilon$. [from (1)]

$[\because O = (O \setminus E) \cup E]$

<u>Case II</u> If $m^*E = \infty$

Take $E_n = E \cap [-n, n]$

then E_n's are measurable and $m^*(E_n) \leq 2n < \infty$

So by above case, $\exists$ an open set $O_n \supset E_n$ s.t.

$m^*(O_n \setminus E_n) < \frac{\epsilon}{2^n}$ — (2)

Take $O = \bigcup_{n=1}^{\infty} O_n$ then O is open & $O \supset E$

$O \setminus E = \bigcup_{n=1}^{\infty} O_n \setminus \bigcup_{n=1}^{\infty} E_n$ $\left(\because O_n \supset E_n \Rightarrow \bigcup_{n=1}^{\infty} O_n \supset \bigcup_{n=1}^{\infty} E_n\right)$

$\subset \bigcup_{n=1}^{\infty} (O_n \setminus E_n)$

$\Rightarrow m^*(O \setminus E) \leq m^*\left(\bigcup_{n=1}^{\infty} O_n \setminus E_n\right) \leq \sum_{n=1}^{\infty} m^*(O_n \setminus E_n)$

$< \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$

$\Rightarrow m^*(O \setminus E) < \epsilon$ <u>Proved</u>.

(III) $\Rightarrow$ (IV) Take $\epsilon = \frac{1}{n}$.

Then $\exists$ an open set $O_n \supset E$ s.t.

$$m^*(O_n \setminus E) < \frac{1}{n}.$$

Define $G = \bigcap_{n=1}^{\infty} O_n$.

Then G is a G_δ set such that $G \supset E$ and

$$m^*(G \setminus E) = m^*\left(\bigcap_{n=1}^{\infty} O_n \setminus E\right)$$
$$\leq m^*(O_n \setminus E) < \frac{1}{n} \quad \forall\, n \in \mathbb{N}$$

taking $n \to \infty$, we get
$m^*(G \setminus E) \leq 0$
but $m^*(G \setminus E) \geq 0$
Hence $m^*(G \setminus E) = 0$

(IV) $\Rightarrow$ (I) we can write
$\cancel{E = G \cap (G \cap E^c)^c}$ $\quad \because E \subset G$
$E = G - (G - E)$
$m^*(G - E) = 0 \Rightarrow G - E$ is measurable
and $G \in \mathcal{M}$ as G is borel set.
$\Rightarrow E \in \mathcal{M}$.

(I) $\Rightarrow$ (II) $E \in \mathcal{M} \Rightarrow E^c \in \mathcal{M}$
So $\exists$ open set $O \supset E^c$ such that
$m^*(O \setminus E^c) < \epsilon$
as $O \supset E^c \Rightarrow O^c \subset E$

and as O is open $\Rightarrow O^c$ is closed
take $F = O^c$
then $E \setminus F = E \cap F^c = E \cap O = O \cap E = O \cap (E^c)^c = O \setminus E^c$
$\Rightarrow m^*(E \setminus F) = m^*(O \setminus E^c) < \epsilon$.

(III) $\Rightarrow$ (V) take $\epsilon = \frac{1}{n}$

then $\exists$ a closed set $F_n \subset E$ s.t.

$$m^*(E - F_n) < \frac{1}{n}$$

take $F = \bigcup_{n=1}^{\infty} F_n$

then F is a F_σ set s.t. $F \subset E$ and

$$m^*(E \setminus F) = m^*\left(E \setminus \bigcup_{n=1}^{\infty} F_n\right) \le m^*(E - F_n) \le \frac{1}{n} \ \forall n \in \mathbb{N}$$

taking $n \to \infty$

$$m^*(E \setminus F) \le 0$$

also $m^*(E \setminus F) \geq 0$

Hence $m^*(E \setminus F) = 0$.

(V) $\Rightarrow$ (I) Write $E^c = F^c - (F^c - E^c)$ $\left[\because F \subset E \Rightarrow F^c \supset E^c\right.$

$F^c - E^c = F^c \cap E = E \cap F^c = E \setminus F$

$\Rightarrow m^*(F^c - E^c) = m^*(E \setminus F) = 0$

$\Rightarrow F^c - E^c \in \mathcal{M}$

also $F^c \in \mathcal{M}$ $\because F \in \mathcal{M}$ $(\because F \in F_\sigma)$

Hence $E^c \in \mathcal{M} \Rightarrow E \in \mathcal{M}$

~~For (VI) $\Rightarrow$ (I) AND (I) $\Rightarrow$ (VI), WE FIRSTLY NEED FOLLOWING RESULT.~~

Problem If $A, B \in \mathcal{M}$ then

$$m^* A + m^* B = m^*(A \cup B) + m^*(A \cap B)$$

Sol. As $A \in \mathcal{M}$

$\Rightarrow m^*(A \cup B) = m^*(A \cup B \cap A) + m^*(A \cup B \cap A^c)$

$= m^*(A) + m^*(A \cup B \cap A^c)$ — (1)

Also $m^* B = m^*(B \cap A) + m^*(B \cap A^c)$

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(I) $\Rightarrow$ (VI)

$$m^* E < \infty \quad (\text{Given})$$

let $\epsilon > 0$ be given. Then $\exists$ a countable collection $\{I_n\}$ of open intervals s.t. $E \subset \bigcup_n I_n$ and

$$\sum \ell(I_n) < m^* E + \frac{\epsilon}{3} \quad \text{— } (*)$$

Then consider $(\bigcup I_n) \setminus E$ ~~is also~~

$\exists$ a countable collection $\{J_n\}$ of open intervals s.t.

$$\bigcup J_n \supset (\bigcup I_n) \setminus E \text{ and}$$

$$\sum \ell(J_n) < m^*\left(\bigcup_n I_n \setminus E\right) + \epsilon/3$$

$$< \epsilon/3 + \epsilon/3 \qquad \left[\because \text{by } (*) \; m^*(\bigcup I_n) - m(E) < \epsilon/3 \Rightarrow m^*(\bigcup I_n \setminus E) < \epsilon/3\right]$$

$$= \frac{2\epsilon}{3} \quad \text{— } (**)$$

Now as $\sum_n \ell(I_n) < m^* E + \epsilon/3 < \infty$, — $(***)$

$\exists$ N s.t. $\displaystyle\sum_{n=N+1}^{\infty} \ell(I_n) < \epsilon/3$

take $U = \displaystyle\bigcup_{n=1}^{N} I_n$

then $E - U \subseteq \bigcup_n I_n - \bigcup_{n=1}^{N} I_n = \bigcup_{n=N+1}^{\infty} I_n$

$\Rightarrow m^*(E \setminus U) \le m^*\left(\bigcup_{n=N+1}^{\infty} I_n\right)$

$$\le \sum_{n=N+1}^{\infty} \ell(I_n) < \epsilon/3 \quad (\text{from } ***)$$

Now $U \setminus E \subseteq (\bigcup I_n) \setminus E \subset \bigcup J_n$

$\Rightarrow m^*(U \setminus E) \le \sum_n \ell(J_n) < 2\epsilon/3$ (from $**$)

$\Rightarrow m^*(E \setminus U) + m^*(U \setminus E) < \epsilon/3 + 2\epsilon/3 = \epsilon$

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$\Rightarrow m^*(U \Delta E) < \epsilon$ $\quad [\because (E\setminus U) \cap (U \setminus E) = \phi]$

(VI) $\Rightarrow$ (II) Given $m^*(U \Delta E) < \epsilon$

$\Rightarrow m^*(U\setminus E) < \epsilon/3$ and $m^*(E\setminus U) < \epsilon/3$.

Now $\exists$ a countable collection $\{I_n\}$ of open intervals s.t.

$E\setminus U \subset \bigcup I_n$ and

$$\sum l(I_n) < m^*(E\setminus U) + \epsilon/3 \qquad \text{—— (a)}$$
$$< \epsilon/3 + \epsilon/3 = 2\epsilon/3$$

Now let $O = U \cup (\bigcup_n I_n)$

$\supset U \cup (E\setminus U) \supset E$

Then O is open and

$$m^*(O\setminus E) = m^*\big((U \cup (\bigcup I_n))\setminus E\big)$$
$$= m^*\big((U\setminus E) \cup (\bigcup I_n)\setminus E\big)$$
$$\le m^*(U\setminus E) + m^*(\bigcup I_n\setminus E)$$
$$< \epsilon/3 + m^*(\bigcup I_n)$$
$$< \epsilon/3 + \sum l(I_n) < \epsilon/3 + 2\epsilon/3 \quad |\text{from (a)}$$

$\Rightarrow m^*(O\setminus E) < \epsilon.$

Exercise Prove that if $E \in \mathcal{M}$, then $E+y \in \mathcal{M}$ $\forall y \in \mathbb{R}$

Proof. let $A \subseteq \mathbb{R}$

then $m^* A = m^*(A\cap E) + m^*(A\cap E^c)$

$\Rightarrow m^*(A+y) = m^*((A\cap E)+y) + m^*((A\cap E^c)+y)$

$[\because m^*$ is translation invariant$]$

Now $(A \cap E) + y = (A+y) \cap (E+y)$

and $(A \cap E^c) + y = (A+y) \cap (E^c + y)$

Hence

$$m^*(A+y) = m^*[(A+y) \cap (E+y)] + m^*[(A+y) \cap (E^c+y)]$$

replacing A by $A-y$, we get

$$m^* A = m^*(A \cap (E+y)) + m^*(A \cap E^c + y)$$

$$= m^*(A \cap (E+y)) + m^*(A \cap (E+y)^c)$$

$\Rightarrow \quad E+y \in \mathcal{M}$.

Proved.

A NON-MEASURABLE SET

for $x, y \in [0,1)$
define

$$x \dot{+} y = \begin{cases} x+y & \text{if } x+y < 1 \\ x+y-1 & \text{if } x+y \geq 1 \end{cases}$$

Similarly for $E \subseteq [0,1)$

$$E \dot{+} y = \{z : z = x \dot{+} y \text{ for some } x \in E\}.$$

Lemma let $E \subset [0,1)$ be a measurable set. Then for each $y \in [0,1)$, the set $E \dot{+} y$ is measurable and $m(E \dot{+} y) = mE$

Proof. let $E_1 = E \cap [0, 1-y)$ and $E_2 = E \cap [1-y, 1)$

Then $E_1 \cup E_2 = E$ and $E_1 \cap E_2 = \phi$

also $E_1, E_2 \in \mathcal{M}$

Now $E_1 \dot{+} y = E_1 + y$
and $E_2 \dot{+} y = E_2 + (y-1)$

$\Rightarrow$ $E_1 \dot{+} y$ and $E_2 \dot{+} y$ are measurable
and $m(E_1 \dot{+} y) = m(E_1 + y) = mE_1$

$m(E_2 \dot{+} y) = m(E_2 + (y-1)) = mE_2$.

As $E \dot{+} y = (E_1 \dot{+} y) \cup (E_2 \dot{+} y)$

and $E_1 \cap E_2 = \phi \Rightarrow (E_1 \dot{+} y) \cap (E_2 \dot{+} y) = \phi$

$\Rightarrow$ $E \dot{+} y \in \mathcal{M}$ and

$$m(E \dot{+} y) = m(E_1 \dot{+} y) + m(E_2 \dot{+} y)$$
$$= mE_1 + mE_2$$
$$= mE.$$

* Construction of a Non-measurable set

for $x, y \in [0,1)$
if $x - y \in \mathbb{Q}$, we say that x and y are equivalent

i.e. $x \sim y$ iff $x - y \in \mathbb{Q}$.

$\sim$ divides $[0,1)$ into equivalence classes.

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If P_1 and P_2 are two different classes, then
$z_1 - z_2 \in \mathbb{R}\setminus\mathbb{Q}$ $\forall\ z_1 \in P_1$ & $\forall z_2 \in P_2$

Let P be the set which contains exactly one element from each equivalence class.

As $\mathbb{Q}\cap[0,1)$ is Countable, let $\langle r_i \rangle_{i=0}^{\infty}$ be an enumeration of rational numbers in $[0,1)$.
where $r_0 = 0$.

Define $P_i = P \dot{+} r_i$

Clearly $P_0 = P$

~~let x~~

claim $P_i \cap P_j = \phi$ for $i \neq j$

let $x \in P_i \cap P_j$ where $i \neq j$

$\Rightarrow$ $x = p_i \dot{+} r_i = p_j \dot{+} r_j$ for some $p_i \in P$ and $p_j \in P$

$\Rightarrow$ $p_i - p_j = r_j - r_i \in \mathbb{Q}$.

$\longrightarrow \leftarrow$.

$\because$ P contains exactly one element from each equivalence class.

$\Rightarrow$ $\langle P_i \rangle$ is a pairwise disjoint sequence of sets.

Conversely if $x \in [0,1)$

$\Rightarrow$ $x \in$ some equivalence class

$\Rightarrow$ $x \sim$ an element in P

$\Rightarrow$ x - (an element in P) $\in \mathbb{Q}$

$\Rightarrow$ $x \in P_i$ for some i.

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Hence $\cup P_i = [0,1)$

Since each P_i is a translation modulo 1 of P,

i.e. $P_i = P \dotplus r_i$

$\Rightarrow P_i \in \mathfrak{M}$ if $P \in \mathfrak{M}$.

and then $m(P_i) = m(P)$

but then
$$m[0,1) = \sum_{i=1}^{\infty} m(P_i) = \sum_{i=1}^{\infty} mP$$

$$\Rightarrow 1 = \sum_{i=1}^{\infty} mP$$

RHS $= 0$ OR ∞ depending on whether

$mP = 0$ or $mP > 0$.

$\longrightarrow\!\leftarrow$.

Hence P is not measurable.

MEASURABLE FUNCTIONS

Proposition 18. Let f be an extended real valued function whose domain is measurable. Then the FAE

(1) for each $\alpha \in \mathbb{R}$, the set $\{x ; f(x) > \alpha\}$ is measurable
(2) for each $\alpha \in \mathbb{R}$, the set $\{x ; f(x) \geq \alpha\}$ is measurable
(3) for each $\alpha \in \mathbb{R}$, the set $\{x ; f(x) < \alpha\}$ is measurable
(4) for each $\alpha \in \mathbb{R}$, the set $\{x ; f(x) \leq \alpha\}$ is measurable.

THESE STATEMENTS IMPLY

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(V) for each extended real number α, the set $\{x;\ f(x) = \alpha\}$ is measurable.

Proof. Let the domain of f be D.

(I) $\Rightarrow$ (IV) $\quad \{x;\ f(x) \le \alpha\} = D - \{x;\ f(x) > \alpha\}$

As difference of two measurable sets is measurable

$\Rightarrow \{x;\ f(x) \le \alpha\} \in \mathcal{M}$.

(IV) $\Rightarrow$ (I) $\quad \{x;\ f(x) > \alpha\} = D - \{x;\ f(x) \le \alpha\}$

(Difference of two measurable sets)

$\Rightarrow \{x;\ f(x) > \alpha\} \in \mathcal{M}$.

Similarly (II) $\Leftrightarrow$ (III)

(I) $\Rightarrow$ (II) $\quad \{x;\ f(x) \geq \alpha\} = \bigcap_{n=1}^{\infty} \{x;\ f(x) > \alpha - \frac{1}{n}\}$

= Intersection of a sequence of measurable sets.

$\Rightarrow \{x;\ f(x) \geq \alpha\} \in \mathcal{M}$.

(II) $\Rightarrow$ (I) $\quad \{x;\ f(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x;\ f(x) \geq \alpha + \frac{1}{n}\}$

= Union of a sequence of measurable sets

$\Rightarrow \{x;\ f(x) > \alpha\} \in \mathcal{M}$.

Hence (I) $\Rightarrow$ (II) $\Rightarrow$ (III) $\Rightarrow$ (IV) $\Rightarrow$ (I)

Now if $\alpha \in \mathbb{R}$

$\{x;\ f(x) = \alpha\} = \{x;\ f(x) \geq \alpha\} \cap \{x;\ f(x) \le \alpha\}$

= intersection of measurable sets

$\Rightarrow \{x;\ f(x) = \alpha\} \in \mathcal{M}$.

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So (II) and (IV) imply (V) for α real.

Since $\{x; f(x) = \infty\} = \bigcap_{n=1}^{\infty} \{x; f(x) \geq n\}$

So (II) $\Rightarrow$ (V) for $\alpha = \infty$.

And as $\{x; f(x) = -\infty\} = \bigcap_{n=1}^{\infty} \{x; f(x) \leq -n\}$

So (IV) $\Rightarrow$ (V) for $\alpha = -\infty$.

Definition :- An extended real valued function f is said to be (Lebesgue) measurable if its domain is measurable and if it satisfies one of the first four statements of above proposition 18.

Remark In proposition 18, (V) does not imply any of the four statements.

Example let $f: [0,1] \to [-1,1]$ by

$$f(x) = \begin{cases} x & ; x \in P \\ -x & ; x \notin P \end{cases}$$

where P is non-measurable set.

For $\alpha \in \mathbb{R}^*$ (i.e. extended real number system)

$$\{x; f(x) = \alpha\} = \begin{cases} \{\alpha\} & \alpha \in P \\ \phi & \alpha \in [0,1] \cap P^c \\ \{-\alpha\} & \alpha \in [-1,0), -\alpha \in P \\ \phi & \alpha \in [-1,0), -\alpha \notin P \\ \phi & \alpha \notin [-1,1] \end{cases}$$

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Here set $\{x ; f(x) = \alpha\}$ is measurable but the set $\{x ; f(x) \geq 0\} = P$; which is not measurable.

Example A constant function with measurable domain is measurable.

Sol. let $f : E \to \mathbb{R}$ be defined by
$$f(x) = c \quad \forall \ x \in E \quad , \text{where } E \in \mathcal{M}.$$

then for $\alpha \in \mathbb{R}$

$$\{x ; f(x) > \alpha\} = \begin{cases} E & \text{if } c > \alpha \\ \phi & \text{if } c \leq \alpha . \end{cases}$$

as $E, \phi \in \mathcal{M}$
$\Rightarrow \{x ; f(x) > \alpha\} \in \mathcal{M}$
$\Rightarrow f$ is measurable function.

Proposition 19 let c be a constant and f and g two measurable real-valued functions defined on the same domain. Then the functions $f+c$, cf, $f+g$, $g-f$ and fg are also measurable.
Proof. Also $\frac{f}{g}$ (g vanishes nowhere on E) is measurable.

Proof for $\alpha \in \mathbb{R}$
$$\{x ; (f+c)(x) < \alpha\} = \{x ; f(x) + c < \alpha\}$$
$$= \{x ; f(x) < \alpha - c\}$$
$$\in \mathcal{M} \quad \because f \text{ is measurable}$$
Hence $f+c$ is measurable.

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Similarly we can show that cf is measurable

$$\{x;\ (f+g)(x) < \alpha\} = \{x;\ f(x)+g(x) < \alpha\}$$
$$= \{x;\ f(x) < \alpha - g(x)\}$$

$\Rightarrow \exists$ a rational no. r s.t.
$$f(x) < r < \alpha - g(x)$$

$$\{x;\ (f+g)(x) < \alpha\} = \bigcup_{r} \left(\{x;\ f(x) < r\} \cap \{x;\ g(x) < \alpha - r\}\right)$$
$$\in \mathcal{M}$$

Hence $f+g$ is measurable.

As $-g = (-1)g$
$\Rightarrow -g$ is also measurable.
$\Rightarrow f-g$ is measurable.

Now for $\alpha \geq 0$ $\{x;\ f^2(x) > \alpha\} = \{x;\ f(x) > \sqrt{\alpha}\} \cup \{x;\ f(x) < -\sqrt{\alpha}\}$

and for $\alpha < 0$
$$\{x;\ f^2(x) > \alpha\} = D$$

So f^2 is measurable.

As $fg = \frac{1}{2}\left[(f+g)^2 - f^2 - g^2\right]$

$\Rightarrow fg$ is measurable.

$f^2(x) > \alpha$, $\alpha \geq 0$
$\Rightarrow f^2(x) - \alpha > 0$
$\Rightarrow (f(x) - \sqrt{\alpha})(f(x) + \sqrt{\alpha}) > 0$
$f(x) - \sqrt{\alpha} > 0$, $f(x) + \sqrt{\alpha} > 0$
or $f(x) - \sqrt{\alpha} < 0$, $f(x) + \sqrt{\alpha} < 0$
$\Rightarrow f(x) > \sqrt{\alpha}$ or
$f(x) < -\sqrt{\alpha}$
$\Rightarrow \{x;\ f(x) > \sqrt{\alpha}\} \cup$
$\{x;\ f(x) < -\sqrt{\alpha}\}$

As $g(x) \neq 0$ $\forall\ x \in E$

$\Rightarrow \frac{1}{g}$ exists.

and $\left\{x ; \frac{1}{g}(x) > \alpha\right\} = \begin{cases} \{x ; g(x) > 0\} & \text{if } \alpha = 0 \\ \{x ; g(x) > 0\} \cap \{x ; g(x) < \frac{1}{\alpha}\} & \text{if } \alpha > 0 \\ \left[\{x ; g(x) < 0\} \cap \{x ; g(x) < \frac{1}{\alpha}\}\right] \cup \{x ; g(x) > 0\} & \text{if } \alpha < 0. \end{cases}$

$\Rightarrow \frac{1}{g}$ is measurable.

Hence $\frac{f}{g}$ is measurable.

* $\overline{\lim}$ and $\underline{\lim}$ of a sequence of functions.

- let $\{x_n\}$ be a real sequence. Then define $y_n = \sup\{x_n, x_{n+1}, \ldots\}$

then $y_1 \geq y_2 \geq y_3 \geq \ldots$ [$\because$ Supremum of bigger set is bigger]

let $y = \inf\limits_{n \geq 1} y_n$

then $y = \overline{\lim\limits_{n \geq 1}}\, x_n = \inf\limits_{n \geq 1} \sup\limits_{k \geq n} x_k$

- Similarly if $z_n = \inf\{x_n, x_{n+1}, \ldots\}$

then $z_1 \leq z_2 \leq z_3 \leq \ldots$ [$\because$ infimum of bigger set is smaller]

let $z = \sup\limits_{n \geq 1} z_n$

then $z = \underline{\lim\limits_{n \geq 1}}\, x_n = \sup\limits_{n \geq 1} \inf\limits_{k \geq n} x_k$

- If a sequence converges, then the three limits $\underline{\lim}$, $\overline{\lim}$ and limit of a sequence are equal.
- $\overline{\lim}$ and $\underline{\lim}$ exist even if sequence does not converge

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- $\overline{\lim}$, $\underline{\lim}$ and supremum and infimum of a set are different.

E.g. Consider $\{-200, 100, 1, 2, -1, 2, -1, 1, 2, -1\}$

then $\sup y_n = \sup \{x_n, x_{n+1}, \ldots\}$

$\Rightarrow y_1 = 100, \quad y_2 = 100, \; y_3 = 2, \; y_4 = 2, \ldots$

$\Rightarrow \overline{\lim} = \inf\limits_{n \geq 1} y_n = 2.$

Similarly $z_n = \inf \{x_n, x_{n+1}, \ldots\}$

$\Rightarrow z_1 = -200, \quad z_2 = -1, \quad z_3 = -1, \ldots$

$\Rightarrow \underline{\lim} = \sup\limits_{n \geq 1} z_n = -1$

but Supremum and infimum of set are 100 and -200 respectively.

[lim Superior tells how large can tails of the sequence eventually be?].

Theorem let $\langle f_n \rangle$ be a sequence of measurable functions (with the same domain of definition). Then the functions $\sup \{f_1, f_2, \ldots, f_n\}$, $\inf \{f_1, f_2, \ldots, f_n\}$, $\sup\limits_n f_n$, $\inf\limits_n f_n$, $\overline{\lim} f_n$ and $\underline{\lim} f_n$ are all measurable.

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Proof (i) Define $h(x) = \sup\{f_1(x), f_2(x), \dots, f_n(x)\}$
let $\alpha \in \mathbb{R}$
then $\{x; h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x; f_i(x) > \alpha\}$

As each f_i is measurable $\Rightarrow$ h is also measurable

similarly define $k(x) = \inf\{f_1(x), f_2(x), \dots, f_n(x)\}$

then $\{x; k(x) > \alpha\} = \bigcap_{i=1}^{n} \{x; f_i(x) > \alpha\}$

$\Rightarrow k$ is measurable.

(ii) let $g = \sup_n f_n$.

then $g(x) = \sup_n f_n(x)$

for $\alpha \in \mathbb{R}$
$\{x; g(x) > \alpha\} = \bigcup_{n=1}^{\infty} \{x; f_n(x) > \alpha\}$

similarly if $g'(\cancel{x}) = \inf_n f_n$

then $\{x; g'(x) > \alpha\} = \bigcap_{n=1}^{\infty} \{x; f_n(x) > \alpha\}$

As countable union and intersection of measurable sets is measurable. So measurability of f_i's implies g and g' are measurable.

(iii) $(\overline{\lim}\, f_n)(x) = \left(\inf_{n \geq 1} \sup_{k \geq n} f_k\right)(x)$

$(\underline{\lim}\, f_n)(x) = \left(\sup_{n \geq 1} \inf_{k \geq n} f_k\right)(x)$

by part (i) and (ii) $\overline{\lim}\, f_n$ and $\underline{\lim}\, f_n$ are measurable

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Exercise If f is continuous function defined on a measurable domain D, then f is measurable

Proof

for $\alpha \in \mathbb{R}$

$$\{x ; f(x) > \alpha\} = f^{-1}(\alpha, \infty)$$

(α, ∞) is open and as f is continuous, so $f^{-1}(\alpha, \infty)$ is also open

$\Rightarrow f^{-1}(\alpha, \infty)$ is measurable.

Converse may not be true.

Example

define $f : [0,2] \to \{1, 2\}$ as

$$f(x) = \begin{cases} 1 , & x \in [0,1] = A \text{ (say)} \\ 2, & x \in (1,2] = A^c \end{cases}$$

Then $$\{x ; f(x) > \alpha\} = \begin{cases} \phi , & \alpha \geq 2 \\ A^c & 1 \leq \alpha < 2 \\ [0,2] , & \alpha < 1 \end{cases}$$

Since ϕ, A^c and $[0,2]$ are measurable.

$\Rightarrow f$ is measurable on $[0,2]$

but f is discontinuous at $x = 1$ and hence not continuous in $[0,2]$.

Exercise If f is a measurable function, then $|f|$ is also measurable

Is the converse true?

Sol

$$\{x ; |f(x)| > \alpha\} = \begin{cases} D \text{ if } \alpha < 0 \\ \{x ; f(x) > \alpha\} \cup \{x ; f(x) < -\alpha\} \text{ if } \alpha \geq 0 \end{cases}$$

$(\because |x| > \alpha \Rightarrow x > \alpha \text{ or } x < -\alpha)$

As f is measurable
$\Rightarrow \{x; f(x) > \alpha\}$ and $\{x; f(x) < -\alpha\}$ are measurable.

$\Rightarrow \{x; f(x) > \alpha\} \cup \{x; f(x) < -\alpha\}$ is measurable.

$\Rightarrow \{x; |f(x)| > \alpha\}$ is measurable.
$\Rightarrow |f|$ is measurable.

Converse: May not be true.
let ~~$A \in \mathbb{R}$ ∉ $\mathcal{M}$~~ A be a non measurable subset of $\mathbb{R}$.
define $f: \mathbb{R} \to \mathbb{R}$ as

$$f(x) = \begin{cases} 4 & \text{if } x \in A \\ -4 & \text{if } x \in A^c \end{cases}$$

then $\{x; f(x) > 0\} = A \notin \mathcal{M}$
$\Rightarrow f$ is not measurable
but $|f|(x) = |f(x)| = 4 \ \forall x \in \mathbb{R}$
which is measurable, being a constant function.

Definition: A property is said to hold almost everywhere (a.e.) if the set of points at which it does not hold is a set of measure zero.

Eg. (1) ~~let $D \in \mathcal{M}$~~
~~then~~ let $f: D \to \mathbb{R}$ and $g: D \to \mathbb{R}$
We say $f = g$ a.e if
$m\{x; f(x) \neq g(x)\} = 0$

(2) let $f: \mathbb{R} \to \mathbb{R}$, $g: \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \\ 0 & \text{if } x \in \mathbb{Q} \end{cases}$$

and $g(x) = 1 \ \forall x \in \mathbb{R}$

then $m\{x;\ f(x) \neq g(x)\} = m(\mathbb{Q}) = 0$

$\Rightarrow f = g$ a.e

Definition If $\{f_n\}$ is a sequence of measurable functions we say $f_n \to f$ a.e. if there exists a set E s.t.
$mE = 0$ and $f_n(x) \to f(x)\ \forall x \in D \setminus E$.

Example let $f_n : [0,1] \to \mathbb{R}$ be defined by
$f_n(x) = x^n$

~~then~~ and $f(x) = \begin{cases} 0 & \text{if } 0 \le x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

Then $f_n(x) \to f(x)$ pointwise but not uniformly

If $g(x) = 0\ \forall\ x \in [0,1]$
then $\lim_{n \to \infty} f_n(x) = g(x)$ a.e
$\because f_n(x) \not\to g(x)$ only at $x = 1$ and
$m(\{1\}) = 0$

Proposition If f is a measurable function and $f = g$ (a.e) then g is also measurable.

Proof let $f : D \to \mathbb{R}^*$ and $g : D \to \mathbb{R}^*$
as $f = g$ a.e
$\Rightarrow$ if $E = \{x : f(x) \neq g(x)\}$, then $m(E) = 0$
now for $\alpha \in \mathbb{R}$
$\{x;\ g(x) > \alpha\} = \{x \in E;\ g(x) > \alpha\} \cup \{x \in D \setminus E,\ g(x) > \alpha\}$

$= \{x \in E;\ g(x) > \alpha\} \cup \left(\{x \in D;\ f(x) > \alpha\} \cap D \setminus E\right)$

$\left[\because f(x) = g(x) \text{ at } D \setminus E\right]$

$= E_1 \cup \left(\{x \in D;\ f(x) > \alpha\} \cap D \setminus E\right)$

$E_1 \in \mathcal{M}$ as $E_1 \subseteq E$ and $m^*(E) = 0 \Rightarrow m^*(E_1) = 0$

$\{x \in D;\ f(x) > \alpha\}$: measurable as f is measurable

$D \setminus E \in \mathcal{M}$ as $D, E \in \mathcal{M}$

Hence g is measurable.

<u>Definition</u> let $A \subseteq \mathbb{R}$, we define the Characteristic functio[n] χ_A as follows:

$$\chi_A = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

<u>Proposition</u> χ_A is measurable iff A is measurable

<u>Proof</u>. let $A \in \mathcal{M}$ and $\alpha \in \mathbb{R}$

then $$\{x;\ \chi_A > \alpha\} = \begin{cases} \phi & \text{if } \alpha \geq 1 \\ A & \text{if } 0 \leq \alpha < 1 \\ D & \text{if } \alpha < 0 \end{cases}$$

$\Rightarrow \{x;\ \chi_A > \alpha\} \in \mathcal{M}$

Hence χ_A is measurable

<u>Converse</u> let χ_A be measurable

$\Rightarrow \{x;\ \chi_A > \alpha\} \in \mathcal{M} \quad \forall \alpha \in \mathbb{R}$

take $\alpha = 0$

then $\{x;\ \chi_A > 0\} = A \Rightarrow A \in \mathcal{M}$

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Definition A real valued function ϕ is called simple if it is measurable and assumes only a finite no. of values.

If $\alpha_1, \alpha_2, \dots, \alpha_n$ are distinct finite values that a simple function assumes and

$$A_i = \{x;\ s(x) = \alpha_i\},\quad i = 1, 2, \dots, n$$

then

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i},\quad A_i \cap A_j = \phi \text{ if } i \neq j$$

Thus we can always express a simple function as a linear combination of characteristic functions. This representation of s is called CANONICAL REPRESENTATION.

Note (1) s is simple $\Leftrightarrow$ A_i's are measurable $\forall i$.
(2) Simple function is always measurable.
(3) The representation of s as linear combination of characteristic functions is not unique.
Sum, product and difference of simple fns is simple.

Eg. Let $s(x) = \chi_{[0,1]} - \chi_{[0,2]} + 3\chi_{[1,3]}$

$$= \begin{cases} 1-1 = 0 & 0 \le x < 1 \\ 1-1+3 = 3 & x = 1 \\ -1+3 = 2 & 1 < x \le 2 \\ 3 & 2 < x \le 3 \end{cases}$$

$$= 3\chi_{\{1\}} + 2\chi_{(1,2]} + 3\chi_{(2,3]}$$

So representation of $s(x)$ is not unique (note that A_i's are not disjoint)

But CANONICAL REPRESENTATION IS ALWAYS UNIQUE.

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Step function :- A real valued function S defined on an interval $[a,b]$ is said to be a step function if there is a partition $a = x_0 < x_1 < \ldots < x_n = b$ such that the function assumes one and only one value in each interval.

Note Step function also assumes finite no. of values like simple functions but the sets $\{x;\ S(x) = c_i\}$ are intervals for each i.

Question Every step function is also a simple fn. but the converse is not true.

Sol. Define $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ 1 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

then f is simple fn. but not step fn. as $\mathbb{Q}$ and $\mathbb{R}\setminus\mathbb{Q}$ are not intervals.

Exercise $\times$ The sum and product of two simple functions is simple.

Proof. Let f, g be two simple functions.

Then

$$f = \sum_{i=1}^{m} \alpha_i \chi_{A_i}, \quad g = \sum_{j=1}^{m'} \beta_j \cdot \chi_{B_j}$$

where $A_i = \{x;\ f(x) = \alpha_i\}$ and $B_j = \{x;\ g(x) = \beta_j\}$

also

$$A_i \cap A_{i'} = \phi \quad \text{for } i \neq i'$$
$$B_j \cap B_{j'} = \phi \quad \text{for } j \neq j'$$

Consider sets E_k obtained by taking $A_i \cap B_j \ \forall (i,j)$

then $f = \sum_{k=1}^{n} \alpha_k \chi_{E_k}$ and $g = \sum_{k=1}^{n} \beta_k \chi_{E_k}$

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then $f+g = \sum_{k=1}^{n} (\alpha_k + \beta_k) \chi_{E_k}$.

where $E_i \cap E_j = \phi$ for $i \neq j$.

$fg = \sum_{k=1}^{n} (\alpha_k \beta_k) \chi_{E_K}$

Hence $f+g$ and fg are also simple.

<u>Theorem</u> If $\langle f_n \rangle$ is a sequence of measurable function on a measurable set E and if $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. on E, then show that f is also measurable on E.

<u>Proof.</u> Let $F = \{x \in E ; \lim_{n\to\infty} f_n(x) \neq f(x)\}$

then $mF = 0$

define g_n and g on E as follows

$$g_n(x) = \begin{cases} 0 & \text{if } x \in F \\ f_n(x) & \text{if } x \notin F \end{cases} \quad , g(x) = \begin{cases} 0 & \text{if } x \in F \\ f(x) & \text{if } x \notin F \end{cases}$$

Then g_n is measurable fn. $\forall n \in \mathbb{N}$.

$x \in F \Rightarrow \lim_{n\to\infty} g_n(x) = 0 = g(x)$

and $x \notin F \Rightarrow \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} f_n(x) = f(x) = g(x)$

$\Rightarrow g_n \to g$ pointwise on E.

as g_n is measurable $\Rightarrow$ g is also measurable.

$\Rightarrow f$ is measurable.

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Littlewood's Three Principles

1. Every measurable set is almost a finite union of intervals.

Proof. Page 26, Proposition 15 (VI condition)

2. Every measurable function is almost continuous.

Proposition let f be a measurable fn. defined on an interval $[a,b]$ and takes values $\pm\infty$ only on a set of measure zero. Then given $\epsilon > 0$, we can find a step function g and a continuous function h such that

$|f-g| < \epsilon$ and $|f-h| < \epsilon$, except on a set of measure less than ϵ.

Further if $m \le f \le M$, then we may choose g and h such that $m \le g \le M$, $m \le h \le M$

Proof Step 1 T.P. $\exists$ M s.t. $|f(x)| \le M$ except on a set of measure less than $\epsilon/3$.

$$\{x;\ f(x) = \pm\infty\} = \bigcap_{n=1}^{\infty} \{x;\ |f(x)| > n\}$$

let $A_n = \{x;\ |f(x)| > n\}$

then $A_1 \supset A_2 \supset A_3 \supset \cdots$

as f is measurable, A_n is a sequence of measurable sets and

$m A_1 \le b-a < \infty$ $[\because A_1 \subseteq [a,b]]$

$$\Rightarrow \lim_{n\to\infty} m A_n = m\left(\bigcap_{n=1}^{\infty} A_n\right) = 0$$

$[\because f(x)$ takes values $\pm\infty$ only on a set of measure zero$]$

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$\Rightarrow$ Given $\epsilon > 0$, $\exists$ M s.t. $m A_M < \epsilon/3$

[$\because m(A_1) \geq m(A_2) \geq \dots$ $\Rightarrow \{m(A_n)\}$ is decreasing seq. bdd below by $0 \Rightarrow$ it cgs to $0 \Rightarrow |m A_M - 0| < \epsilon/3$]

and $|f(x)| < M$ except on A_M.

Step II To prove Given $\epsilon > 0$, $\exists$ a simple function s.t.

$$|f(x) - \phi(x)| < \epsilon \text{ except on } A_M.$$

Define for $n = 1, 2, 3, \dots$

$$E_k = \left\{ x ; \frac{(k-1)M}{n} \leq f(x) \leq \frac{kM}{n} \right\} \text{ for } 1-n \leq k \leq n$$

[i.e. equivalent to say $|f(x)| \leq M$]

as f is measurable $\Rightarrow E_k \in \mathcal{M} \ \forall k$

Now define $\phi_n = \sum_{k=1-n}^{n} \frac{(k-1)}{n} M \chi_{E_k}$

as $E_k \in \mathcal{M} \Rightarrow \phi_n$ is measurable
~~Also~~ So ϕ_n is simple fn.

and $|f - \phi_n| \leq \frac{kM}{n} - \frac{(k-1)M}{n} = \frac{M}{n}$

[$\because$ if $x \in E_i \Rightarrow f(x) \leq \frac{iM}{n}$ and $\phi_n(x) = \frac{(i-1)M}{n}$]

Given $\epsilon > 0$ Choose N st $\frac{M}{N} < \epsilon$

then $|f - \phi_N| < \epsilon$
take $\phi = \phi_N$

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then $|f-\phi| < \epsilon$ when $|f| < M$

and $|f(x)| < M$ except on A_M

$\Rightarrow |f-\phi| < \epsilon$ except on A_M.

Step III T.P. There exists a step function g on $[a,b]$ s.t $g(x) = \phi(x)$ except on a set of measure less than $\epsilon/3$.

let $\phi(x) = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ be canonical representation of ϕ.

Then as $A_i \in \mathcal{M}$ $\forall i$ and $m^*(A_i) < \infty$ ($\because$ we are in finite interval $[a,b]$)

$\Rightarrow \exists$ a finite union U_i of open intervals s.t $m^*(A_i \Delta U_i) < \frac{\epsilon}{3^n}$ (by first Principle)

Define $g(x) = \sum_{i=1}^{n} \alpha_i \chi_{U_i}$

Then g is a step fn as U_i's are intervals.

and $g(x) = \phi(x)$ except on $\bigcup_{i=1}^{n} (A_i \Delta U_i)$

and $m^*\left(\bigcup_{i=1}^{n} (A_i \Delta U_i)\right) \le \sum_{i=1}^{n} m^*(A_i \Delta U_i)$

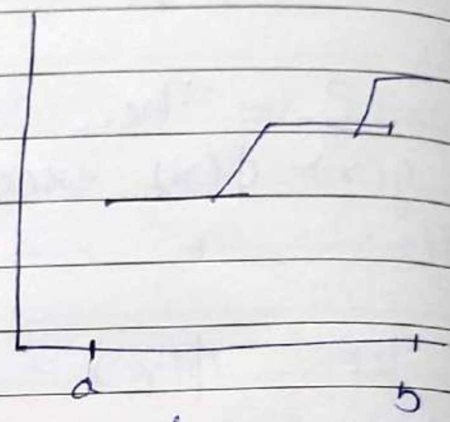
$$< \sum_{i=1}^{n} \frac{\epsilon}{3^n} = \frac{\epsilon}{3} < \epsilon$$

Thus $g(x) = \phi(x)$ except on a set of measure less than $\epsilon/3$.

Pf.

Step IV Given a step fn. g on $[a,b]$ there is a cont. fn h s.t. $g(x) = h(x)$ except on a set of measure less than $\epsilon/3$.

On each of the intervals (a_i, b_i) (or (a_i, b_i) or $(a_i, b_i]$ or $[a_i, b_i]$) of which $[a,b]$ is made of.



Define $h(x) = g(x)$ except on $\left(a_i, a_i + \frac{\epsilon}{6n}\right)$, $\left(b_i - \frac{\epsilon}{6n}, b_i\right)$ where we define $h(x)$ by linearity (a straight line)

Then $g(x) = h(x)$ except on a set of measure $\sum_{i=1}^{n} \frac{\epsilon}{3n} = \frac{\epsilon}{3} < \epsilon$

$\therefore$ by step II and step III

$|f - h| < \epsilon$ except on a set of measure less than ϵ.

3. Every convergent sequence of measurable functions is nearly uniformly convergent

Proposition Let E be a measurable set of finite measure, let $\{f_n\}$ be a sequence of measurable functions that converges to a real valued fn. f a.e. on E. Then

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given $\epsilon > 0$ and $\delta > 0$ there is a set $A \subset E$ with $mA < \delta$ and an N s.t.

$|f_n(x) - f(x)| < \epsilon \quad \forall\ x \in E \setminus A$ and $\forall\ n \geq N$.

(i.e. except at A, $f_n \to f$)

Proof. let F be a set of measure zero s.t.
$f_n(x) \to f(x) \quad \forall \quad x \in E \setminus F$. (pointwise)

Let $G_n = \{x \in E;\ |f_n(x) - f(x)| \geq \epsilon\}$

and $E_N = \bigcup_{n=N}^{\infty} G_n = \{x \in E;\ |f_n(x) - f(x)| \geq \epsilon\}$ for some $n \geq N$

as f_n's are measurable and $f_n \to f$ a.e. on E, so f is also measurable and so $f_n - f$ is also measurable and so is $|f_n - f|$.

$\Rightarrow G_n \in \mathcal{M} \quad \Rightarrow E_N \in \mathcal{M}$.

Also $E_N \supseteq E_{N+1} \quad \forall \quad N$

Also $\bigcap_{N=1}^{\infty} E_N \subseteq F$.

$\because$ if $x \notin F \Rightarrow f_n(x) \to f(x)$

$\Rightarrow \exists\ N$ s.t. $|f_n(x) - f(x)| < \epsilon \quad \forall\ n \geq N$

$\Rightarrow x \in G_n \quad \forall\ n \geq N$

$\Rightarrow x \notin E_N \Rightarrow x \notin \bigcap_{N=1}^{\infty} E_N$

also $m(E_n) < \infty$ as $mE < \infty$

$\Rightarrow \lim_{N \to \infty} m\, E_N = m\left(\bigcap_{N=1}^{\infty} E_N\right) \leq mF = 0$

$\Rightarrow \lim_{N \to \infty} m\, E_N = 0$

$\Rightarrow$ Given $\delta > 0$, $\exists$ some N_0 s.t.
$m E_N < \delta \quad \forall \; N \geq N_0$.

Taking $A = E_{N_0}$

$mA < \delta$ and also for each $x \in E \setminus A$

$|f_n(x) - f(x)| < \epsilon \quad \forall \; n \geq N_0$.

Remark: The proposition is not true if $mE = \infty$
Consider $E = [1, \infty)$
and take $f_n = \chi_{[1,n]}$; $n = 1, 2, 3, \ldots$

If $f(x) = 1 \quad \forall \; x \in E$
then $f_n(x) \to f(x)$

let $x \geq 1$
then $\exists$ some N s.t. $x \in [1, N+1]$

$\Rightarrow f_{N+1}(x) = 1$

and $f_n(x) = 1 \quad \forall \; n > N$

$f_n(x) = 0 \quad \forall \; n \leq N$

$\Rightarrow \lim_{n \to \infty} f_n(x) = 1$

take $\epsilon = 1$, $\delta = 1$ to find A s.t. $mA < 1$

$|f_n(x) - 1| < \epsilon = 1 \quad \forall \; x \in E \setminus A \quad \forall \; n \geq N$

i.e. $|\chi_{[1,n]} - 1| < 1 \quad \forall \; n \geq N, \; \forall \; x \in E \setminus A$

but $|\chi_{[1,n]} - 1| \nless 1 \quad \forall \; x \in E \setminus [1, n]$

So it is not possible to find A s.t. $mA < 1$

Hence given proposition is not true if $mE = \infty$.

Lebesgue Integration

* Firstly we recall Riemann Integration

Let f be a bounded real valued function defined on the interval $[a,b]$ and let $a = x_0 < x_1 < \dots < x_n = b$ be a sub-division of $[a,b]$.

Then for each sub-division we can define

$$S = \sum_{i=1}^{n} (x_i - x_{i-1}) M_i \quad \text{and} \quad s = \sum_{i=1}^{n} (x_i - x_{i-1}) m_i$$

where $M_i = \sup_{x_{i-1} \leq x \leq x_i} f(x)$ and $m_i = \inf_{x_{i-1} < x \leq x_i} f(x)$

then the upper Riemann integral and lower Riemann Integral of f are defined by

$$R\int_a^{\bar{b}} f(x)\,dx = \inf S \quad \text{and} \quad R\int_{\underline{a}}^{b} f(x)\,dx = \sup s$$

where inf. and sup are taken over all possible subdivisions of $[a,b]$.

Also $\int_{\underline{a}}^{b} f(x)\,dx \leq \int_a^{\bar{b}} f(x)\,dx$ always holds

and if both are equal then $f \in R[a,b]$ and Riemann Integration is denoted by

$$R\int_a^b f(x)\,dx.$$

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Now if we define $\psi(x) = d_i \ \forall \ x \in [x_{i-1}, x_i]$ for some subdivision $[a,b]$ and some set constants d_i, then

$$\int_a^b \psi \, dx = \int_{x_0}^{x_1} \psi(x)\,dx + \int_{x_1}^{x_2} \psi(x)\,dx + \cdots + \int_{x_{n-1}}^{x_n} \psi(x)\,dx$$

$$= \int_{x_0}^{x_1} \alpha_1\,dx + \int_{x_1}^{x_2} d_2\,dx + \cdots + \int_{x_{n-1}}^{x_n} d_n\,dx$$

$$= \sum_{i=1}^{n} d_i (x_i - x_{i-1})$$

So we see that $R\int_a^{\bar{b}} f(x)\,dx = \inf \sum_{i=1}^{n} M_i (x_i - x_{i-1})$

$$= \inf \int_a^b \psi(x)\,dx \quad \text{for all step}$$

functions $\psi(x) \geq f(x)$

Similarly $R\int_{\underline{a}}^{b} f(x)\,dx = \sup \sum_{i=1}^{n} m_i (x_i - x_{i-1})$

$$= \sup \int_a^b \phi(x)\,dx \quad \text{for all step}$$

function $\phi(x) \leq f(x)$.

<u>Note</u>: In this chapter we introduce the concept of Lebesgue Integration by firstly defining Lebesgue Integration of a simple fn. and then we see that the Lebesgue Integration of a bounded measurable fn. on a set of finite measure is

$$\inf_{\substack{\psi(x) \geq f(x) \\ \psi \text{ simple}}} \int \psi(x)\,dx = \sup_{\substack{\phi(x) \leq f(x) \\ \phi \text{ simple}}} \int \phi(x)\,dx$$

and thus it extends the Concept of Riemann Integration.

The only difference in Riemann Integration and Lebesgue Integration is that in Riemann Integration we divide $[a,b]$ in intervals and so we set step functions while in Lebesgue Integration we divide $[a,b]$ or any other domain into measurable sets and so we get simple functions.

But Lebesgue Integration helps us to find integration of some of those fns also which are not Riemann Integrable. On the other hand every Riemann Integrable fn. is Lebesgue Integrable. (we will prove).

Def. Integral of a Simple function

If $\phi = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$ is the simple fn. (RHS being the canonical representation of ϕ. Here α_i's are finite in number, distinct and non-zero and E_i's are disjoint) s.t. ϕ vanishes outside a set of finite measure. Then

$$\int \phi = \sum_{i=1}^{n} \alpha_i \, m A_i .$$

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First fundamental Thm of Calculus

If f is continuous on $[a, b]$ and F is the indefinite integral of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Second fundamental Thm.

If f is continuous on an open interval I and a is any pt in I and if

$$F = \int_a^x f(t)\,dt, \text{ then}$$

$$F'(x) = f(x) \quad \text{at each pt in } I.$$

In this Chapter, we will show that result of second fundamental Thm holds almost everywhere.

The first result is true only for a certain class of functions which we shall characterize.

Differentiation and Integration

<u>Def</u> Let $\mathcal{I}$ be a collection of proper intervals. Let $E \subseteq \mathbb{R}$. Then $\mathcal{I}$ is said to be a Vitali Cover of set E if for each $x \in E$ and each $\epsilon > 0$, $\exists$ an interval $I \in \mathcal{I}$ s.t. $x \in I$ and $l(I) < \epsilon$.

<u>Example</u> let $\{r_n\}$ be enumeration of the rationals in $[a,b]$. Then the Collection $\{I_{n,i}\}$, where

$$I_{n,i} = \left[r_n - \frac{1}{i} , r_n + \frac{1}{i} \right], \quad n, i \in \mathbb{N}$$ forms a Vitali cover of $[a,b]$.

<u>Note</u> The intervals in $\mathcal{I}$ may be open, closed or half open. But we do not allow degenerate intervals of only one point.

<u>Vitali Covering Lemma</u> Let E be a set of finite outer measure and $\mathcal{I}$ be a collection of intervals that covers E in the sense of Vitali. Given $\epsilon > 0$, $\exists$ a finite disjoint collection $\{I_1, I_2, \ldots, I_N\}$ of intervals in $\mathcal{I}$ such that

$$m^*\left(E \setminus \bigcup_{n=1}^{N} I_n \right) < \epsilon$$

<u>Proof</u> If E is contained in a finite disjoint union of members of $\mathcal{I}$ say $I_1, I_2, \ldots, I_N$, then

$$E \setminus \bigcup_{n=1}^{N} I_n = \phi$$ and hence nothing to prove.

Let E be not contained in a finite disjoint union of members of $\mathcal{I}$ W.L.O.G. assume that each interval of $\mathcal{I}$ is closed for otherwise we could replace interval by its closure and note that the set of end points of $I_1, I_2, \ldots, I_N$ has measure zero.

Now as $m^*(E) < \infty$, $\exists$ an open set $O \supseteq E$ s.t.
$$m^*(O \setminus E) < \epsilon$$

$$\Rightarrow m^*(O) < m^*(E) + \epsilon < \infty$$
$\Rightarrow$ O has finite outer measure.

Consider the subfamily $\mathcal{I}' = \{I \in \mathcal{I} : I \subseteq O\}$

<u>claim</u> $\mathcal{I}'$ is also a vitali cover of E.
for this let $x \in E$ and $\epsilon > 0$ be given. then as $E \subseteq O$ and O is open
$\exists$ $\epsilon' > 0$ s.t.
$$(x-\epsilon', x+\epsilon') \subseteq O$$
let $\epsilon_1 = \min\{\epsilon, \epsilon'\}$
then by def. of Vitali Cover, $\exists$ $I \in \mathcal{I}$ s.t.
$$l(I) < \epsilon_1 \quad \text{and} \quad x \in I$$
$$\Rightarrow I \subset (x-\epsilon_1, x+\epsilon_1)$$
$$\subseteq (x-\epsilon', x+\epsilon')$$
$$\subseteq O$$
$$\Rightarrow I \subseteq O. \quad \Rightarrow I \in \mathcal{I}'$$

choose a sequence $\{I_n\}$ of disjoint intervals of $\mathcal{I}'$ by induction on n.
let $I_1 \in \mathcal{I}'$ be arbitrary
and suppose $I_1, I_2, \ldots, I_n$ have already been chosen.

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Take $k_n = \sup \{ l(I);\ I \in \mathcal{G}';\ I \cap (\bigcup_{j=1}^{n} I_j) = \phi \}$ — (A)

Now since $I \in \mathcal{G}'$ $\Rightarrow$ $I \subseteq O$ $\Rightarrow$ $l(I) < m^*(O) < \infty$.

Then by def. of supremum, $\exists$ some $I_{n+1} \in \mathcal{G}'$ s.t.

$\frac{k_n}{2} < l(I_{n+1}) < k_n$ (If we take $\epsilon = \frac{k_n}{2}$)

(B)

and $I_{n+1} \cap (\bigcup_{j=1}^{n} I_j) = \phi$

as each $I_n \subseteq O \Rightarrow \bigcup_{n=1}^{\infty} I_n \subseteq O$

$\Rightarrow \sum_{n=1}^{\infty} l(I_n) \leq m^*(O) < \infty$.

$\Rightarrow \exists\ N$ s.t. $\sum_{n=N+1}^{\infty} l(I_n) < \frac{\epsilon}{5}$ [$\because$ series $\sum_{n=1}^{\infty} l(I_n)$ Converges $\Rightarrow$ seq. of partial sum converges]

let $R = E \setminus \bigcup_{n=1}^{N} I_n$.

Claim $m^*(R) < \epsilon$

let $x \in R \Rightarrow x \notin \bigcup_{n=1}^{N} I_n$ (which is a closed set)

~~But $\mathcal{G}'$ is vitali cover of E~~

$\Rightarrow \exists\ I \in \mathcal{G}'$ s.t. $x \in I$ and $I \cap (\bigcup_{n=1}^{N} I_n) = \phi$.

[we can choose this way at top]

If $I \cap I_i = \phi$ for $1 \leq i \leq n$

$\Rightarrow l(I) \leq k_n < 2 l(I_{n+1})$ (by A and B)

also as $\sum_n l(I_n)$ is cgt $\Rightarrow \lim_{n \to \infty} l(I_{n+1}) = 0$

⇒ I must intersect atleast one of the intervals

~~(as I is proper~~
(∵ otherwise $\ell(I) < 2\ell(I_{n+1}) \ \forall n$
$\Rightarrow \ell(I) = 0$ $\left(\because \lim\limits_{n\to\infty} \ell(I_{n+1}) = 0\right)$

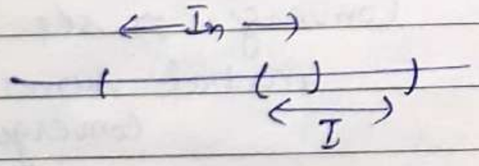
⇒ I is singleton)

let n be the smallest integer s.t. $I \cap I_n = \phi$
Then $n > N$ (by our choice)

and $\ell(I) < k_{n-1} < 2\ell(I_n)$

since $x \in I$ and $I \cap I_n \neq \phi$
⇒ Distance of x from mid pt of I_n

$$\le \ell(I) + \frac{1}{2}\ell(I_n)$$



$$< 2\ell(I_n) + \frac{1}{2}\ell(I_n)$$

$$= \frac{5}{2}\ell(I_n)$$

⇒ $x \in J_n$, where J_n is an interval having same mid point as I_n and has five times larger length.

then $R \subseteq \bigcup\limits_{n=N+1}^{\infty} J_n$

$$\Rightarrow m^*(R) \le \sum_{n=N+1}^{\infty} \ell(J_n) = \sum_{n=N+1}^{\infty} 5\ell(I_n) < \frac{\epsilon}{5}\cdot 5$$

$$\Rightarrow m^*\left(E \setminus \bigcup_{n=1}^{N} I_n\right) < \epsilon$$

Hence the result.

we define $D^+ f(x) = \overline{\lim}_{h \to 0^+} \dfrac{f(x+h) - f(x)}{h}$.

$$D^- f(x) = \overline{\lim}_{h \to 0^+} \frac{f(x) - f(x-h)}{h}$$

$$D_+ f(x) = \underline{\lim}_{h \to 0^+} \frac{f(x+h) - f(x)}{h}$$

$$D_- f(x) = \underline{\lim}_{h \to 0^+} \frac{f(x) - f(x-h)}{h}.$$

clearly $D^+ f(x) \geq D_+ f(x)$ and $D^- f(x) \geq D_- f(x)$.

If $D^+ f(x) = D_+ f(x) = D^- f(x) = D_- f(x) \neq \pm\infty$, we say that f is differentiable at x and define $f'(x)$ to be the common value of derivatives at x.

If $D^+ f(x) = D_+ f(x) \Rightarrow f$ has RHS derivative at x and $f'(x+)$ to be their common value.

Similarly for $f'(x-)$.

~~Proposition If f is continuous on $[a, b]$ and one of its derivatives (say D^+) is everywhere non-negative on (a, b), then f is non decreasing on $[a, b]$ i.e. $f(x) \leq f(y)$ for $x \leq y$.~~

Theorem* Let f be an increasing real-valued function on the interval $[a, b]$. Then f is differentiable almost everywhere. The derivative f' is measurable, and

$$\int_a^b f'(x)\,dx \leq f(b) - f(a).$$

Proof. firstly we show that the sets where any two derivatives are unequal, have measure zero.

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let $E = \{x \in [a,b] : D^+f(x) > D_-f(x)\}$

The sets arising from other combination of derivatives being handled similarly.

let $E_{u,v} = \{x : D^+f(x) > u > v > D_-f(x)\}$
for all rationals u and v.

and $E = \bigcup_{u,v \in Q} E_{u,v}$.

<u>To Show</u> $m^*(E) = 0$ It is enough to show that $m^* E_{u,v} = 0 \ \forall u,v \in Q$.

let $s = m^* E_{u,v}$
Then given $\epsilon > 0$, $\exists$ an open set $O \supset E_{u,v}$ s.t.

$m^*(O) < m^* E_{u,v} + \epsilon = s + \epsilon$

Now if $x \in E_{u,v} \subset O$, then

$D_- f(x) < v$

$\Rightarrow \sup_{\delta > 0} \inf_{0<h<\delta} \dfrac{f(x) - f(x-h)}{h} < v$

$\Rightarrow \forall \delta > 0, \exists h, 0 < h < \delta$

s.t. $f(x) - f(x-h) < vh$. —— (1)

$\Rightarrow \forall x \in E_{u,v}, \exists$ an interval $[x-h, x] \subset O$
s.t.
$f(x) - f(x-h) < vh$

So $\{[x-h, x] ; x \in E_{u,v}\}$ is a Vitali Covering of $E_{u,v}$ [$\because$ length of $[x-h, x]$ ~~Ea~~ is small by ~~taking~~ h to be small]

$$A = B \cap C$$
$$m^*(B) = m^*(A) + m^*(B \setminus C)$$

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So $\exists$ finitely many disjoint intervals $\{I_1, \dots, I_N\}$ each contained in O s.t.

$$m^*\left(E_{u,v} \setminus \bigcup_{n=1}^{N} I_n\right) < \epsilon \quad \text{— (2)} \quad \text{(By Vitali Covering ~~Thm~~ lemma)}$$

Write $I_n = [x_n - h_n, x_n]$, $x_n \in E_{u,v}$

Then $$\sum_{n=1}^{N} [f(x_n) - f(x_n - h_n)] < v \sum_{n=1}^{N} h_n \quad \text{— by (1)}$$

$$\left[\because \sum_{n=1}^{N} h_n = \sum_{n=1}^{N} l(I_n) = l\left(\bigcup_{n=1}^{N} I_n\right) \leq m(O) \right.$$

$< v\, m(O) < v(s + \epsilon)$

$\left.\because I_n\text{'s are disjoint}\right]$

Let $A = E_{u,v} \cap \left(\bigcup_{n=1}^{N} I_n^{\circ}\right)$

Then $s = m^*(E_{u,v}) \leq m^*(A) + m^*(E_{u,v} \setminus \bigcup I_n^{\circ})$

$\left[\because E_{u,v} = A \cup (E_{u,v} \setminus \bigcup I_n^{\circ})\right]$

$\Rightarrow s \leq m^*(A) + \epsilon$ (by (2))

$\Rightarrow m^*(A) > s - \epsilon$ — (3)

let $y \in A$

$\Rightarrow y \in E_{u,v}$ also $\Rightarrow D^+ f(y) > u$

$\Rightarrow$ as earlier, $\exists$ an arbitrary small interval $[y, y+k] \subset I_i^{\circ}$ for some $i = 1, 2, \dots, N$

such that $f(y+k) - f(y) > uk$

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$\Rightarrow$ we get a Vitali cover $\mathcal{I}^*$ of A.

By Vitali Covering lemma, $\exists$ finite disjoint coll $\{J_1, J_2, \ldots, J_M\}$ s.t.

$$m^*\left(A \setminus \bigcup_{j=1}^{M} J_j\right) < \epsilon.$$

but $m^*(A) > s - \epsilon$

$$\Rightarrow m^*\left(A \cap \bigcup_{j=1}^{M} J_j\right) > s - 2\epsilon.$$

$\Big[\because A = (A \cap \cup J_j) \cup (A \setminus \cup J_j)$

$\Rightarrow m^*(A) \le m^*(A \cap \cup J_j) + m^*(A \setminus \cup J_j)$

$\Rightarrow m^*(A \cap \cup J_j) \geq m^*(A) - m^*(A \setminus \cup J_j)$

$> s - \epsilon - \epsilon = s - 2\epsilon \Big]$

Also each $J_j = [y_j, y_j + k_j]$, $j = 1, 2, \ldots, M$

$$\sum_{j=1}^{M} [f(y_j + k_j) - f(y_j)] > u \sum_{j=1}^{M} k_j$$

$$> u\, m^*(A) > u(s - \epsilon)$$

but each J_j is contained in some I_i

$\Rightarrow$ for a fixed I_i, summing over those j for which $J_j \subset I_i$, we get

$$\sum f(y_j + k_j) - f(y_j) \le f(x_i) - f(x_i - h_i)$$

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As f is increasing, so summing RHS over $i = 1, 2, \ldots, N$, we get

$$\sum_{i=1}^{N} [f(x_i) - f(x_i - h_i)] \geq \sum_{j=1}^{M} [f(y_j + k_j) - f(y_j)]$$

$\Rightarrow v(s+\epsilon) > u(s - 2\epsilon)$

which holds for each $\epsilon > 0$

$\Rightarrow vs \geq us$

but $u > v \Rightarrow s = 0$

$\Rightarrow m^* E = 0$.

Hence $g(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ is defined a.e. in $[a, b]$ and f is differentiable whenever g is finite.

Now let $g_n(x) = \dfrac{f\left(x + \frac{1}{n}\right) - f(x)}{1/n}$

where $f(x) = f(b)$ for $x \geq b$

Then $\{g_n\}$ is a seq. of non-neg. fns.

As f is increasing, $g_n \to g$ a.e.

Also g is measurable

$\Rightarrow$ By Fatou's lemma,

$$\int_a^b g \leq \underline{\lim}_{n \to \infty} \int_a^b g_n$$

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$$= \lim_{n\to\infty} n \int_a^b \left[f\left(x+\frac{1}{n}\right) - f(x) \right] dx$$

$$= \lim_{n\to\infty} n \left[\int_b^{b+\frac{1}{n}} f - \int_a^{a+\frac{1}{n}} f \right]$$

$$= \lim n \left[f(b)\cdot\frac{1}{n} - \int_a^{a+\frac{1}{n}} f \right] \qquad \left[\because f(x) = f(b) \ \forall x \geq b \right]$$

$$= \lim f(b) - \lim n \int_a^{a+\frac{1}{n}} f. \quad (1)$$

$$\int_a^{a+\frac{1}{n}} f(x)\,dx \geq f(a)\cdot\frac{1}{n} \qquad \left[\because f(x) \geq f(a) \text{ if } x \in \left[a, a+\frac{1}{n}\right] \right]$$

$$\Rightarrow n \int_a^{a+\frac{1}{n}} f(x)\,dx \geq f(a)$$

from (1)

$$\int_a^b g \leq f(b) - f(a) < \infty.$$

$\Rightarrow$ g is integrable and is finite a.e.

Hence f is diff. a.e. and

$f' = g$ a.e.

$\Rightarrow$ f' is measurable.

and $\int f' \leq f(b) - f(a)$.

Functions of Bounded Variation

let $f: [a,b] \to \mathbb{R}$ and let $a = x_0 < x_1 < \dots < x_k = b$ be any subdivision of $[a,b]$.

Define

$$p = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^+$$

$$n = \sum_{i=1}^{k} [f(x_i) - f(x_{i-1})]^-$$

$$t = n + p = \sum_{i=1}^{k} |f(x_i) - f(x_{i-1})|$$

and clearly $p - n = f(b) - f(a)$

let $P = \text{Sup}\, p$, $N = \text{Sup}\, n$, $T = \text{Sup}\, t$

where we take the suprema over all possible subdivisions of $[a,b]$.

Clearly $P \le T \le P + N$

P, N and T are called the positive, negative and total variations of f over $[a,b]$.

T_a^b or $T_a^b(f)$ is written to denote the dependence of the interval $[a,b]$ or on the function f.

If $T < \infty$, we say that f is of BOUNDED VARIATION over $[a,b]$.

also written as $f \in BV$

Examples 1. A bounded monotone function is a function of bounded variation.

and $T_a^b(f) = |f(b) - f(a)|$

2. A function of bounded variation is bounded but the converse is not true.

Proof. let $f: [a,b] \to \mathbb{R}$ be a fn. of bounded variation.

T.P. f is bounded.

$$\cancel{|f(x) - f(a)| \leq} \; |f(x)| - |f(a)| \leq |f(x) - f(a)|$$

$$\Rightarrow |f(x)| \leq |f(a)| + |f(x) - f(a)|$$

$$\Rightarrow |f(x)| \leq |f(a)| + \sup_{x_0 < x_1 < \dots < x_n} |f(x_i) - f(x_{i-1})|$$

$\Rightarrow f(x)$ is bounded.

Example of a bounded fn. which is not of bounded variation.

Consider $f: [0,1] \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x \sin\left(\frac{\pi}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

clearly f is bounded
as $-1 \leq \left(\sin \frac{\pi}{x}\right) \leq 1$
and $x \in [0,1]$ $\Rightarrow$ $-1 \leq x \sin \frac{\pi}{x} \leq 1$.

Claim: f is not of bounded variation.

Consider partition $\left\{1, \frac{2}{3}, \frac{2}{5}, \frac{2}{7}, \ldots, \frac{2}{2n+1}, 0\right\}$ of $[0,1]$ where $n \in \mathbb{N}$.

then $T_0^1(f) \geq \left|f(1) - f\left(\frac{2}{3}\right)\right| + \left|f\left(\frac{2}{3}\right) - f\left(\frac{2}{5}\right)\right| + \cdots + \left|f\left(\frac{2}{2n+1}\right) - f(0)\right|$

$$= \left|0 + \frac{2}{3}\right| + \left|\frac{2}{3} + \frac{2}{5}\right| + \cdots + \left|\frac{2}{2n+1} - 0\right|$$

$$= 4\left(\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2n+1}\right)$$

$$= \cancel{4\sum_n \frac{1}{2n+1}} \quad \text{~~which~~}$$

but $\sum \frac{1}{2n+1}$ is divergent and so its partial sum $\frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots + \frac{1}{2n+1}$ is not bounded above

$\Rightarrow T_0^1 f = \infty$.

* Example of a Continuous fn. which is not of bounded variation.

Same example as above will work

$\because \lim_{x \to 0} x \sin \frac{\pi}{x} = 0$ $\left[\because \lim_{x\to 0} f(x)g(x) = 0 \text{ if } \lim_{x\to 0} f(x) = 0 \text{ and } g(x) \text{ is a bounded fn}\right.$

Lemma: If f is of bounded variation on $[a,b]$, then

$$T_a^b = P_a^b + N_a^b$$

and

$$f(b) - f(a) = P_a^b - N_a^b.$$

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Proof. for any subdivision of $[a,b]$

$$t = p + n \quad \text{—(1)}$$

and $f(b) - f(a) = p - n \quad$ —(2)

(1) + (2) and (1) − (2) gives

$t + f(b) - f(a) = 2p$ and $t - f(b) + f(a) = 2n$

$\Rightarrow$ $t = 2p - f(b) + f(a)$ and $t = 2n + f(b) - f(a)$

Taking suprema over all possible sub-divisions of $[a,b]$.

$$T = 2P + f(a) - f(b) \quad \text{— (3)}$$

and $T = 2N + f(b) - f(a) \quad$ — (4)

(3) + (4) $\Rightarrow$ $2T = 2P + 2N$
$\Rightarrow T = P + N$

(3) − (4) $\Rightarrow$ $0 = 2(P - N) + 2\,(f(a) - f(b))$
$\Rightarrow P - N = f(b) - f(a)$.

Jordan Decomposition Theorem A fn. f is of bounded variation on $[a,b]$ if and only if f is the difference of two monotone real valued functions on $[a,b]$.

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Proof: let $f: [a,b] \to R$

firstly suppose that f is a function of bounded variation

Then $f = g - (g - f)$

Claim $\exists$ g and $g-f$ are non-decreasing fns on $[a,b]$.

as $f = g - (g - f)$

$\Rightarrow f(x) = g(x) - (g(x) - f(x))$, $x \in [a,b]$.

Now let $x, y \in [a,b]$ s.t. $x < y$.

Then $T_a^y(f) = T_a^x(f) + T_x^y(f)$

$\Rightarrow T_a^y(f) - T_a^x(f) = T_x^y(f)$

$\Rightarrow g(y) - g(x) = T_x^y(f)$

where $g(x) = T_a^x(f)$.

and $g(y) - g(x) \geq 0$

$\Rightarrow g(y) \geq g(x)$.

$\Rightarrow$ g is non-decreasing fn. on $[a,b]$.

Now we prove that $g-f$ is non-decreasing on $[a,b]$.

let $x, y \in [a,b]$ s.t. $x < y$

then as above $g(y) - g(x) = T_x^y(f)$

but $I_x^y(f) \geq |f(y) - f(x)|$

$\Rightarrow g(y) - g(x) \geq |f(y) - f(x)|$
$\geq f(y) - f(x)$

$\Rightarrow (g-f)(y) \geq (g-f)(x)$

$\Rightarrow g-f$ is non-decreasing on $[a,b]$.

Conversely let $g(x)$ and $h(x)$ be increasing functions s.t.

$$f(x) = g(x) - h(x)$$

TP $\quad f \in BV[a,b]$

let $P = \{a = x_0 < x_1 < x_2 < \dots < x_n = b\}$ be the sub-division of $[a,b]$.

and let $T = \sum_{r=0}^{n-1} |f(x_{r+1}) - f(x_r)|$

then $|f(x_{r+1}) - f(x_r)| = |(g-h)(x_{r+1}) - (g-h)(x_r)|$

$= |(g(x_{r+1}) - g(x_r)) - (h(x_{r+1}) - h(x_r))|$

$\leq |g(x_{r+1}) - g(x_r)| + |h(x_{r+1}) - h(x_r)|$

as g and h are monotonically increasing

$\Rightarrow g(x_{r+1}) - g(x_r) \geq 0$ and $h(x_{r+1}) - h(x_r) \geq 0$.

$\Rightarrow |g(x_{r+1}) - g(x_r)| = g(x_{r+1}) - g(x_r)$ and
$|h(x_{r+1}) - h(x_r)| = h(x_{r+1}) - h(x_r)$

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$\Rightarrow |f(x_{i+1}) - f(x_i)| \le (g(x_{i+1}) - g(x_i)) + (h(x_{i+1}) - h(x_i))$

$\Rightarrow \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| \le g(b) - g(a) + h(b) - h(a)$

as f is finite in $[a,b]$

$\Rightarrow g(b), g(a), h(b), h(a)$ are all finite no.

$\Rightarrow \sum_{i=0}^{n-1} |f(x_{i+1}) - f(x_i)| < \infty$

$\Rightarrow T_a^b(f) < \infty$

$\Rightarrow f \in BV[a,b]$

Hence proved.

Corollary 6 If f is of bounded variation on $[a,b]$, then $f'(x)$ exists for almost all x in $[a,b]$

Proof. As f is of bounded variation

$\Rightarrow f(x) = g(x) - h(x)$, where g and h monotonic non-decreasing functions.

Also we know that a monotonic increasing fn. is always differentiable a.e.

$\Rightarrow g'(x), h'(x)$ exist a.e.

$\Rightarrow f'(x) = g'(x) - h'(x)$ exists a.e.

(by Theorem next to Vitali Covering Lemma)

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Differentiation of an Integral

If f is an integrable fn. defined on $[a,b]$, we define its indefinite integral to be the function F defined on $[a,b]$ by

$$F(x) = \int_a^x f(t)\,dt.$$

We will show that the derivative of the indefinite integral of an integrable fn is equal to the integrand a.e.

<u>Lemma 7</u> If f is integrable on $[a,b]$, then the fn. F defined by

$$F(x) = \int_a^x f(t)\,dt$$

is a continuous function of bounded variation on $[a,b]$.

<u>Proof</u>. let $x_0 \in [a,b]$

Then
$$|F(x) - F(x_0)| = \left| \int_a^x f(t)dt - \int_a^{x_0} f(t)dt \right|$$

$$= \left| \cancel{\int_a^x f(t)\,dt + \int_{x_0}^a f(t)\,dt} \right|$$

$$= \left| \int_a^{x_0} f(t)\,dt + \int_{x_0}^x f(t)dt - \int_a^{x_0} f(t)\,dt \right|$$

$$= \left| \int_{x_0}^x f(t)\,dt \right|$$

$$\leq \int^{x} |f(t)|\,dt$$

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But f is integrable on $[a,b]$
$\Rightarrow |f|$ is integrable on $[a,b]$.

$\Rightarrow$ Given $\epsilon > 0$, $\exists\ \delta > 0$ s.t. for every measurable set $A \subset [a,b]$ with $m(A) < \delta$, we have

$$\int_A |f| < \epsilon .$$

[**Prop.** Let f be a non-neg. fn. which is integrable over a set E. Then given $\epsilon > 0$, $\exists\ \delta > 0$ s.t. for every set $A \subset E$ with $mA < \delta$, we have

$$\int_A f < \epsilon .$$

Proof. Case 1 If f is bounded.
Then $\exists\ K > 0$ s.t.

$$|f(x)| \leq K \quad \forall\ x \in E .$$

$\Rightarrow$ for given $\epsilon > 0$, $\exists\ \delta\ (= \epsilon / K) > 0$ s.t.

$$\int_A f = f \cdot m(A)$$

but $|f| \leq K \Rightarrow \int_A f \leq K m(A) < K\delta = K \cdot \frac{\epsilon}{K} = \epsilon .$

Case 2 If f is unbounded on E
Consider a seq. of functions $\{f_n\}$ on E given by

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n \\ n & \text{if } f(x) > n . \end{cases}$$

Then $\{f_n\}$ is an increasing seq. of non-neg. bounded measurable functions on E s.t.

$$\lim_{n \to \infty} f_n = f \quad \text{on } E .$$

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So by Monotone Convergence theorem, ~~given $\epsilon>0$,~~ ~~there is an integer N s.t~~

$$\int_E f_N > \int_E f - \epsilon/2$$

$$\lim_{n\to\infty} \int f_n = \int f$$

$\Rightarrow$ given $\epsilon > 0$, $\exists$ an integer N s.t.

$$\int_E f_N > \int_E f - \epsilon/2$$

Claim

$$\int_E (f-f_N) = \int_E f - \int_E f_N$$

$$\int_E f = \int_E (f-f_N+f_N) = \int_E f-f_N + \int_E f_N \quad -(1)$$

[$\because$ $f-f_N$ and f_N are non-neg. and measurable]

further f being integrable over E,

$$\int_E f < \infty$$

$\Rightarrow$ each integral on the RHS of (1) is finite.

In particular $\int_E f_N < \infty$

So from (1)

$$\int_E f-f_N = \int_E f - \int_E f_N$$

Since $\int_E f_N > \int_E f - \epsilon/2$

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$$\Rightarrow \int_E f_N - \int_E f > -\frac{\epsilon}{2}$$

$$\Rightarrow \int_E f - \int_E f_N < \epsilon/2$$

$$\Rightarrow \int_E (f - f_N) < \epsilon/2$$

Choose $\delta < \frac{\epsilon}{2N}$

then if $m(A) < \delta$, then

$$\int_A f = \int_A (f - f_N) + \int_A f_N$$

$$\leq \int_E (f - f_N) + N \cdot m(A)$$

$$< \epsilon/2 + N \frac{\epsilon}{2N} = \epsilon$$

$$\Rightarrow \int_A f < \epsilon. \Big]$$

$$\Rightarrow \left| \int_{x_0}^{x} |f(t)|\, dt \right| < \epsilon \quad \text{for } |x - x_0| < \delta$$

$$\Rightarrow |F(x) - F(x_0)| = \left| \int_{x_0}^{x} f(t)\, dt \right| \leq \int_{x_0}^{x} |f(t)|\, dt \,\Big| < \epsilon$$

whenever $|x - x_0| < \delta$

$\Rightarrow F$ is continuous at x_0 and hence in $[a, b]$.

Now to prove F is a function of bounded variation let $P = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a, b]$. Then

$$\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| = \sum_{i=1}^{n} \left| \int_{x_{i-1}}^{x_i} f(t)\, dt \right|$$

$$\leq \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} |f(t)|\, dt$$

$$= \int_a^b |f(t)|\, dt.$$

$$\Rightarrow T_a^b(F) \leq \int_a^b |f(t)|\, dt.$$

but as $|f|$ is integrable,

$$\Rightarrow \int_a^b |f(t)|\, dt < \infty$$

$$\Rightarrow T_a^b(F) < \infty$$

$$\Rightarrow F \in BV[a,b].$$

Lemma: If f is integrable on $[a,b]$ and

$$\int_a^x f(t)\, dt = 0 \quad \forall\, x \in [a,b]$$

then $f(t) = 0$ a.e. in $[a,b]$.

Proof: let $f \neq 0$ a.e. in $[a,b]$

let $f(t) > 0$ on a set E of positive measure. Then $\exists$ a closed set $F \subset E$ with $m(F) > 0$

let $O = (a,b) - F$

then O is open.

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Now $\int_a^b f(t)\,dt = \int_{O\cup F} f(t)\,dt$

but $\int_a^b f(t)\,dt = 0$ (given)

$\Rightarrow \int_{O\cup F} f(t)\,dt = 0$

$\Rightarrow \int_O f(t)\,dt + \int_F f(t)\,dt = 0$

$\Rightarrow \int_O f(t)\,dt = -\int_F f(t)\,dt$

but $f(t) > 0$ on F with $mF > 0$ implies

$\int_F f(t)\,dt \neq 0$

$\Rightarrow \int_O f(t)\,dt \neq 0$

but O being an open set, can be expressed as a union of countable collection $\{(a_n, b_n)\}$ of disjoint open intervals as we know that an open set can be expressed as a union of countable collection of disjoint open intervals.

$\Rightarrow \int_O f(t)\,dt = \sum_n \int_{a_n}^{b_n} f(t)\,dt$

$\Rightarrow \sum_n \int_{a_n}^{b_n} f(t)\,dt \neq 0 \quad \left[\because \int_O f(t)\,dt \neq 0\right]$

$\Rightarrow \int_{a_n}^{b_n} f(t)\,dt \neq 0$ for some n.

$\Rightarrow$ either $\int_{a}^{a_n} f(t)\,dt \neq 0$

or $\int_{a}^{b_n} f(t)\,dt \neq 0$

In either case, we see that if f is +ve on a set of positive measure, then for some $x \in [a,b]$, we have

$$\int_a^x f(t)\,dt \neq 0.$$

Similarly, if f is -ve on a set of +ve measure we have

$$\int_a^x f(t)\,dt \neq 0.$$

$\longrightarrow \longleftarrow$

$\Rightarrow f = 0$ a.e. in $[a,b]$

Lemma 9 If f is bounded and measurable on $[a,b]$ and

$$F(x) = \int_a^x f(t)\,dt + F(a)$$

then $F'(x) = f(x)$ for almost all x in $[a,b]$

Proof. By lemma 7, F is a fn. of bounded variation and so by Cor 6, $F'(x)$ exists for almost all x in $[a,b]$.

Now f is bounded.

let $|f| \leq K$.

Consider $\dfrac{F(x+h)-F(x)}{h} = f_n(x)$ where $h=\dfrac{1}{n}$

$$\Rightarrow f_n(x) = \frac{1}{h}\left[\int_a^{x+h} f(t)\,dt + F(a) - \int_a^x f(t)\,dt - F(a)\right]$$

$$= \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

$$\Rightarrow |f_n(x)| \le \frac{1}{h}\int_x^{x+h} |f(t)|\,dt$$

$$\le \frac{K}{h}\int_x^{x+h} dt = K.$$

$$\Rightarrow |f_n| \le K$$

and $f_n(x) \to F'(x)$ a.e.

So $\{f_n\}$ is a seq. of bounded measurable fns s.t. $\underset{n\to\infty}{\text{Lt}}\, f_n(x) = F'(x)$ a.e.

So by Bounded Convergence thm.,

$$\int_a^x F'(x)\,dx = \lim_{n\to\infty}\int_a^x f_n(x)\,dx$$

$$= \lim_{h\to 0}\frac{1}{h}\int_a^x \big(F(x+h)-F(x)\big)\,dx \quad \left[\because h=\frac{1}{n}\right]$$

$$= \lim_{h\to 0}\frac{1}{h}\int_a^x F(x+h)\,dx - \frac{1}{h}\int_a^x F(x)\,dx.$$

$$= \underset{h\to 0}{\text{Lt}}\ \frac{1}{h}\int_{a+h}^{x+h} F(x)\,dx - \frac{1}{h}\int_a^x F(x)\,dx$$

$$= \underset{h\to 0}{\text{Lt}}\ \frac{1}{h}\int_{a+h}^{x} \cancel{F(x)\,dx} + \frac{1}{h}\int_x^{x+h} F(x)\,dx - \frac{1}{h}\int_a^{a+h} F(x)\,dx - \frac{1}{h}\int_{a+h}^{x} \cancel{F(x)\,dx}$$

$$= \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} F(x)\,dx - \frac{1}{h}\int_a^{a+h} F(x)\,dx$$

$$= F(x) - F(a) \longrightarrow (*)$$

$$= \int_a^x f(t)\,dt.$$

$$\Rightarrow \int_a^x \left(F'(t) - f(t)\right) dt = 0 \quad \forall\, x \in [a,b]$$

$F'(x) - f(x) = 0$ a.e in $[a,b]$ (by lemmas)

$\Rightarrow F'(x) = f(x)$ a.e in $[a,b]$

~~Theorem 10 Let f be an integrable function on $[a,b]$ and suppose that~~

~~$$F(x) = F(a) + \int_a^x f(t)\,dt$$~~

~~Then $F'(x) = f(x)$ for almost all x in $[a,b]$.~~

Reason of $(*)$ $\lim_{h\to 0} \frac{1}{h}\int_x^{x+h} F(x)\,dx = \lim_{h\to 0} \frac{1}{h}\left[G(x)\right]_x^{x+h}$

$$= \lim_{h\to 0} \frac{G(x+h) - G(x)}{h}$$

$$= G'(x)$$

where $G(x) = \int_x^{x+h} F(x)\,dx$

$\Rightarrow G'(x) = F(x)$

($\because$ fundamental theorem of calculus holds if F is continuous)

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Here f is bounded and measurable on $[a,b]$

$\Rightarrow$ f is integrable on $[a,b]$

$\Rightarrow$ by lemma 7, $F(x) = \int_a^x f(t)\,dt$ is a continuous function.

Theorem 10 Let f be an integrable fn. on $[a,b]$ and suppose that

$$F(x) = F(a) + \int_a^x f(t)\,dt$$

Then $F'(x) = f(x)$ for almost all x in $[a,b]$.

Proof. Without loss of generality we may assume that $f(x) \geq 0$. $\forall x$.

[$\because$ we can write $f = f^+ - f^- = g - h$ (say)

So if $G(x) = G(a) + \int_a^x g(t)\,dt \Rightarrow G'(x) = g(t)$ a.e.

$F(x) = G(x) - H(x)$

and $H(x) = H(a) + \int_a^x h(t)\,dt \Rightarrow H'(x) = h(t)$ a.e.

then $F'(x) = g(t) - h(t) = f(t)$ a.e.]

Define a sequence $\{f_n\}$ of functions $f_n : [a,b] \to R$ where

$$f_n(x) = \begin{cases} f(x) & \text{if } f(x) \leq n \\ n & \text{if } f(x) > n. \end{cases}$$

Clearly each f_n is a bounded and measurable fn.

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and so by previous theorem,

$$\frac{d}{dx}\int_a^x f_n = f_n(x) \quad a.e.$$

Also $f - f_n \geq 0 \quad \forall n$.

So the fn G_n defined by

$$G_n(x) = \int_a^x f - f_n$$ is an increasing function of x.

So by thm. (*)

$G_n(x)$ must have a derivative a.e and it must be non-negative.

Now $$G_n(x) = \int_a^x (f - f_n)(t)\,dt$$

$$= \int_a^x f(t)\,dt - \int_a^x f_n(t)\,dt$$

$$\Rightarrow \int_a^x f(t)\,dt = G_n(x) + \int_a^x f_n(t)\,dt.$$

Hence the relation $F(x) = \int_a^x f(t)\,dt + F(a)$ becomes

$$F(x) = G_n(x) + \int_a^x f_n(t)\,dt + F(a) \quad -(1)$$

$$\Rightarrow F'(x) = G_n'(x) + f_n(x) \quad a.e.$$

$$\geq f_n(x) \quad a.e. \quad \forall n$$

Since n is arbitrary, $F'(x) \geq f(x)$ a.e.

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$$\Rightarrow \int_a^b F'(x)\,dx \geq \int_a^b f(x)\,dx \quad —— (2)$$

Since G_n is increasing fn., by (1) $F(x)$ is also increasing fn. of x.
So again by thm (*)

F' is measurable and

$$\int_a^b F'(x)\,dx \leq F(b) - F(a) \quad —— (3)$$

but $$F(x) = \int_a^x f(t)\,dt + F(a)$$

$$\Rightarrow F(b) = \int_a^b f(x)\,dx + F(a)$$

$$\Rightarrow \int_a^b f(x)\,dx = F(b) - F(a)$$

So from (3)

$$\int_a^b F'(x)\,dx \leq \int_a^b f(x)\,dx \quad —— (4)$$

Hence by (3) and (4)

$$\int_a^b F'(x)\,dx = \int_a^b f(x)\,dx. \Rightarrow \int_a^b \left(F'(x) - f(x)\right)dx = 0$$

$$\Rightarrow F'(x) - f(x) = 0 \text{ a.e.} \quad [\because F'(x) - f(x) \geq 0 \text{ a.e.}]$$

$$\Rightarrow F'(x) = f(x) \text{ a.e.}$$

Proved.

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Remark An indefinite integral need not be differentiable everywhere and even if it is differentiable, it need not follow that

$$F' = f \text{ everywhere.}$$

Example Define $f : [0,1] \to R$ by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \\ 0 & \text{otherwise} \end{cases}$$

then $F(x) = \int_0^x f(t)\,dt = \int_{Q \cap [0,x]} f(t)\,dt + \int_{[0,x] \setminus Q} f(t)\,dt$

$= 0 + 0 \qquad [\because m^*(Q) = 0]$

$= 0$

So F defines a function differentiable in $[0,1]$ but

$$F'(x) \neq f(x) \text{ for } x \in [0,1] \cap Q.$$

Absolute Continuity

A real valued function f defined on $[a,b]$ is said to be absolutely continuous on $[a,b]$ if given $\epsilon > 0$ $\exists$ $\delta > 0$ s.t.

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \epsilon$$

seq.

for every ~~finite~~ collection $\{(x_i, x_i')\}$ of nonoverlapping intervals with

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$$\sum_{i=1}^{n} |x_i' - x_i| < \delta.$$

Note (1) Every absolutely Continuous fn. is uniform continuous (obvious from def.) and hence continuous.

(2) Uniformly continuous fn. need not be absolutely Continuous.

Example Cantor fn. in $[0,1]$ (How?) $f(x) = \begin{cases} x \sin\frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

then $f(x)$ is cont. but not absolutely cont. on $[0,1]$

Lemma 11 If f is absolutely continuous on $[a,b]$, then it is of bounded variation on $[a,b]$.

Proof As f is absolutely cont. fn., so given $(1=)\epsilon > 0$, $\exists\ \delta > 0$ s.t.

$$\sum_{i=1}^{n} |f(x_i') - f(x_i)| < \epsilon = 1 \quad (\text{say})$$

for every finite Collection $\{(x_i, x_i')\}$ of non-overlapping intervals with

$$\sum_{i=1}^{n} |x_i' - x_i| < \delta$$

let N be the natural no. s.t.

$$N > \frac{b-a}{\delta}.$$

Then dividing $[a,b]$ as

$$a = c_0 < c_1 < c_2 \dots < c_N = b \text{ s.t. } c_j - c_{j-1} < \delta \ \forall\ j = 1, 2, \dots, N$$

Therefore, for every finite collection $\{(x_i, x_i')\}$ of pairwise disjoint subintervals in $[c_{j-1}, c_j]$,

$$\sum_i f(x_i') - f(x_i) < 1$$

$$\Rightarrow T_{c_{j-1}}^{c_j}(f) \leq 1 \qquad j = 1, 2, \dots, N$$

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$$\Rightarrow T_a^b(f) = \sum_{j=1}^{N} T_{c_{j-1}}^{c_j}(f) \leq N < \infty.$$

$\Rightarrow f \in BV[a,b]$

Cor 12 If f is absolutely cont, then f' exist a.e.

Proof as f is absolutely cont

$\Rightarrow f \in BV[a,b]$

$\Rightarrow f = g - h$ where g and h are some monotonic non-decreasing fun.

$\Rightarrow f' = g' - h'$ a.e.

$\Rightarrow f'$ exist a.e.

lemma 13 If f is absolutely cont ~~and~~ on $[a,b]$ and $f'(x) = 0$ a.e., then f is constant.

Proof Let $c \in [a,b]$ be arbitrary.

claim $f(c) = f(a)$

Let $E \subset (a,c)$ be the set of measure $c-a$ in which $f'(x) = 0$

Let $\epsilon, \eta > 0$ be arbitrary.

Now $f'(x) = 0 \ \forall x \in E$

$\Rightarrow$ there is an arbitrary small interval $[x, x+h]$ s.t.

$$\frac{|f(x+h) - f(x)|}{h} < \eta$$

$$\Rightarrow |f(x+h) - f(x)| < \eta h \qquad (1)$$

$\Rightarrow$ corresponding to every x, $\exists$ an arbitrary small interval $[x, x+h] \subset [a, c]$ s.t.

$$|f(x+h) - f(x)| < \eta h.$$

$\Rightarrow$ Intervals $[x, x+h]$ $\forall x \in E$, cover E in Vitali's sense. Thus by Vitali's Lemma, we can find a finite no. of non-overlapping intervals I_k, where

$I_k = [x_k, y_k]$ $\forall$ $k = 1, 2, \ldots, n$ such that this collection covers all of E except for a set of measure less than $\delta > 0$ where $\delta > 0$ is the no. corresponding to $\epsilon > 0$ in the def. of absolute continuity of f.

Let $x_k < x_{k+1}$, we have

$$a = y_0 \leq x_1 < y_1 \leq x_2 < y_2 < \cdots < y_n \leq x_{n+1} = c$$

As f is absolute continuous, so for above subdivision of $[a, c]$, we have

$$\sum_{k=0}^{n} |f(x_{k+1}) - f(y_k)| < \epsilon \quad \text{whenever} \quad \sum_{k=0}^{n} |x_{k+1} - y_k| < \delta$$

Then from (1)

$$\sum_{k=1}^{n} |f(y_k) - f(x_k)| \leq \eta \sum_{k=1}^{n} (y_k - x_k)$$
$$< \eta (c-a)$$

and

$$|f(c) - f(a)| = \left| \sum_{k=0}^{n} [f(x_{k+1}) - f(y_k)] + \sum_{k=1}^{n} [f(y_k) - f(x_k)] \right|$$
$$\leq \sum_{k=0}^{n} |f(x_{k+1}) - f(y_k)| + \sum_{k=1}^{n} |f(y_k) - f(x_k)|$$
$$< \epsilon + \eta (c-a)$$

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but ϵ, η and hence $\epsilon + \eta$ $(c-a)$ are arbitrary

$\Rightarrow f(c) = f(a)$

Hence proved.

Theorem 14 A function f is an indefinite integral if and only if it is absolutely Continuous.

Proof let $F = \int f(t)\, dt$.

and $\epsilon > 0$ be given.
Then $\exists$ $\delta > 0$ s.t. for every measurable set $A \subset [a,b]$ with $mA < \delta$, we have

! $\int_A |f| < \epsilon$ [f integrable $\Rightarrow$ $|f|$ is also integrable]

[$\because$ if f is non-neg. fn. integrable over a set E, then given $\epsilon > 0$, $\exists$ $\delta > 0$ s.t. for every $A \subset E$ with $mA < \delta$, $\int_A f < \epsilon$].

So for any finite collection $\{(x_i, x_i')\}$; $i=1, \ldots, N\}$ of pairwise disjoint open intervals in $[a, b]$ with

$\sum_{i=1}^{N} (x_i' - x_i) < \delta$, we have

$$\sum_{i=1}^{n} \left| \int_{x_i}^{x_i'} f(t)\, dt \right| \leq \sum_{i=1}^{n} \int_{x_i}^{x_i'} |f(t)|\, dt$$

$$< \epsilon$$

$$\Rightarrow \sum_{i=1}^{N} |F(x_i') - F(x_i)| < \epsilon.$$

$\Rightarrow F$ is absolutely continuous.

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Conversely let F be absolutely continuous on $[a,b]$.
$\Rightarrow$ F is of bounded variation and so we can write

$$F(x) = F_1(x) - F_2(x)$$

where F_1, F_2 are monotone increasing.
$\Rightarrow F'(x)$ exists a.e. and

$$|F'(x)| = |F_1'(x) - F_2'(x)| \le |F_1'(x)| + |F_2'(x)|$$
$$= F_1'(x) + F_2'(x)$$

and $$\int_a^b |F'| \le \int_a^b F_1' + \int_a^b F_2'$$
$$\le F_1(b) - F_1(a) + F_2(b) - F_2(a)$$
$$< \infty$$
(by thm. (x))

$\Rightarrow |F'|$ is integrable on $[a,b]$ and hence F' is integrable on $[a,b]$.

let $$G(x) = \int_a^x F'(t)\,dt.$$

Then by previous part, G is absolutely continuous and so is fn. $F - G = H$ (say).

then $$H'(x) = F'(x) - G'(x) = F'(x) - F'(x) \text{ a.e.}$$
(by thm. 10)

$\Rightarrow$ $H'(x) = 0$ a.e.

$\Rightarrow$ H is Constant by lemma 13

$\Rightarrow$ $F = G + H$

$$\Rightarrow F(x) = \int_a^x F'(t)\,dt + \cancel{F(a)}\ H.$$

taking $x = a$

$F(a) = H$

$$\Rightarrow F(x) = \int_a^x F'(t)\,dt + F(a)$$

Hence proved.

<u>Corollary 15</u>. Every absolutely continuous function is the indefinite integral of its derivative.

<u>Proof</u> Same as above.

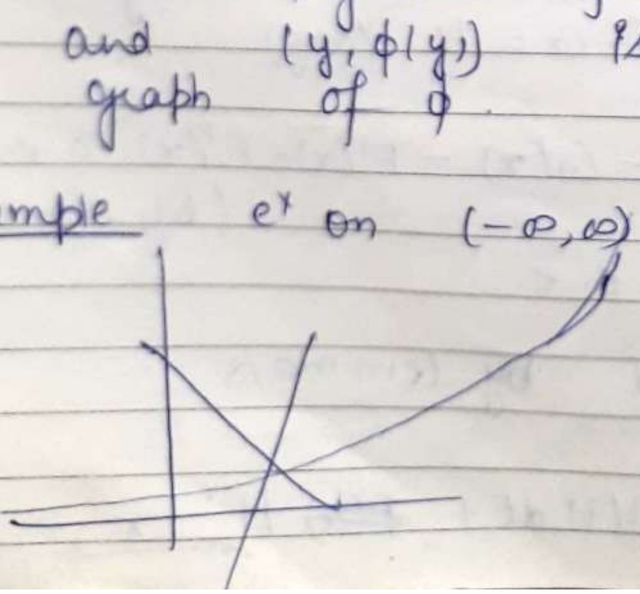
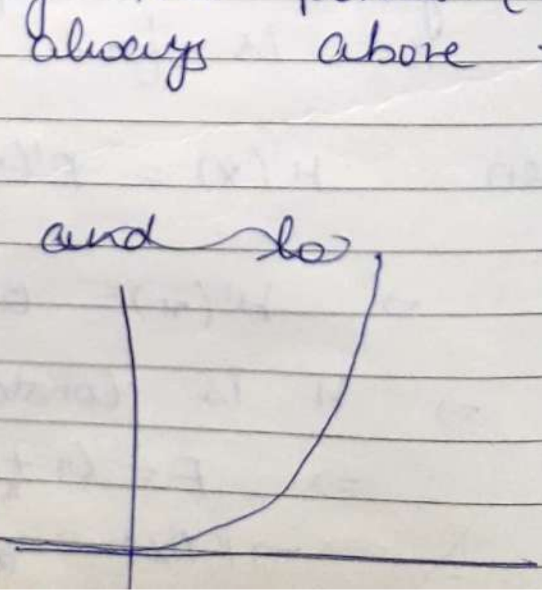
Convex functions

A function ϕ defined on an open interval (a,b) is said to be convex if for each $x, y \in (a,b)$ and each λ, $0 \le \lambda \le 1$ we have

$$\phi(\lambda x + (1-\lambda)y) \le \lambda \phi(x) + (1-\lambda)\phi(y)$$

Geometrically, we can say that ~~chord between~~ line segment joining the points $(x, \phi(x))$ and $(y, \phi(y))$ is always above the graph of ϕ.

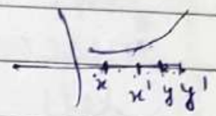
<u>example</u> e^x on $(-\infty, \infty)$ and so,

Lemma 16 If ϕ is convex on (a,b) and if x, y, x', y' are points of (a,b) s.t. $x \le x' < y'$ and $x < y \le y'$ then the chord over (x', y') has larger slope than the chord over (x, y); i.e.

$$\frac{\phi(y) - \phi(x)}{y - x} \le \frac{\phi(y') - \phi(x')}{y' - x'}$$

Proof $\quad x < y \le y'$



then $0 < y - x \le y' - x \Rightarrow 0 < \dfrac{y-x}{y'-x} \le 1$

let $\dfrac{y-x}{y'-x} = \lambda$

then $\quad y - x = \lambda(y' - x)$

$$\Rightarrow y = \lambda y' + (1-\lambda)x \quad \text{and } 0 < \lambda \le 1$$

$$\Rightarrow \phi(y) \le \lambda\phi(y') + (1-\lambda)\phi(x) \quad [\because \phi \text{ is convex}]$$

$$\Rightarrow \phi(y) - \phi(x) \le \lambda\phi(y') - \lambda\phi(x)$$

$$\Rightarrow \phi(y) - \phi(x) \le \lambda(\phi(y') - \phi(x))$$

$$= \frac{y-x}{y'-x}(\phi(y') - \phi(x))$$

$$\Rightarrow \frac{\phi(y) - \phi(x)}{y-x} \le \frac{\phi(y') - \phi(x)}{y'-x} \quad \text{—(1)}$$

Similarly $\quad x \le x' < y'$ implies that

$$\frac{\phi(y') - \phi(x)}{y'-x} \le \frac{\phi(y') - \phi(x')}{y'-x'} \quad \text{—(2)}$$

$\left[\because y'-x \ge y'-x' > 0 \Rightarrow \dfrac{y'-x'}{y'-x} \le 1\right]$

from (1) and (2), we get

$$\frac{\phi(y) - \phi(x)}{y-x} \le \frac{\phi(y') - \phi(x')}{y'-x'}$$

Proved

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Now if $x_0 \in (a,b)$ then

$$\frac{\phi(x)-\phi(x_0)}{x-x_0}$$ is an increasing fn. of x

by previous thm. $\left[\because\ y \le y' \Rightarrow \frac{\phi(y)-\phi(x)}{y-x} \le \frac{\phi(y')-\phi(x)}{y'-x}\right]$

$\Rightarrow \underset{x\to x_0^+}{Lt}$ and $\underset{x\to x_0^-}{Lt}$ exist and are finite.

$\Rightarrow \phi$ is differentiable on right and on left at each point.

and left hand derivative is less than or equal to right hand derivative.

If $x_0 < y_0$, $x < y_0$ and $x_0 < y$ then

$$\frac{\phi(x)-\phi(x_0)}{x-x_0} \le \frac{\phi(y)-\phi(y_0)}{y-y_0}$$

and either derivative at x_0 is less than or equal to either derivative at y_0.

$\Rightarrow$ Each derivative is monotone and so they are equal at a pt if one of them is continuous there.

Since monotone functions have only a countable no of discontinuities, they are equal except on a countable set.

<u>Proposition 18</u> If ϕ is a continuous function on (a,b) and if one derivative (say D^+) of ϕ is non-decreasing, then ϕ is convex.

<u>Proof.</u> Given x, y with $a < x < y < b$, define a function ψ on $[0,1]$ by

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$$\psi(t) = \phi[ty + (1-t)x] - t\phi(y) - (1-t)\phi(x)$$

To prove that ϕ is a convex function, we will show that

$$\psi(t) \le 0 \quad \forall \ t \in [0,1]$$

Now ψ is continuous and $\psi(0) = \psi(1) = 0$

and $D^+\psi = (y-x) D^+\phi - \phi(y) + \phi(x)$

as $(y-x) > 0$ and $D^+\phi$ is non-decreasing

$\Rightarrow D^+\psi$ is also non-decreasing on $[0,1)$

$\left[\because D^+\psi(t_1) - D^+\psi(t_2) = (y-x) D^+\phi(t_1 y + (1-t_1)x) - (y-x) D^+\phi(t_2 y + (1-t_2)x) \right]$

Let r be the point where ψ assumes its maximum on $[0,1]$. [$\because \psi$ is cont so it assumes its max and min in $[0,1]$]

If $r = 1$, then

$$\psi(t) \le \psi(1) = 0 \quad \text{on } [0,1]$$

So assume that $r \in [0,1)$

Since ψ has a local maximum at r

$\Rightarrow D^+\psi(r) \le 0$

but $D^+\psi$ is non-decreasing

$\Rightarrow D^+\psi \le 0$ on $[0,r]$

Consequently ψ is non-increasing on $[0,r]$ and hence $\psi(r) \le \psi(0) = 0$

$\Rightarrow$ max of ψ on $[0,1]$ is non-positive

$\Rightarrow \psi \le 0$ on $[0,1]$ Hence Proved.

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Corollary 19 Let ϕ have a second derivative at each point of (a,b). Then ϕ is convex on (a,b) if and only if $\cancel{\phi^{n}(x) \geq 0}$ $\phi''(x) \geq 0$ for each $x \in (a,b)$

Proof. Let $\phi''(x) \geq 0$ for each $x \in (a,b)$
$\Rightarrow \phi'(x)$ is non-decreasing and as $\phi'(x)$ exists $\Rightarrow \phi$ is continuous also

$\Rightarrow \phi$ is convex (by thm 18)

Now let ϕ be a convex fn. then by prop. 17, $\phi''(x) \geq 0 \quad \forall x \in (a,b)$

Def. Let ϕ be a convex function on (a,b) and $x_0 \in (a,b)$. The line

$$y = m(x - x_0) + \phi(x_0)$$

through $(x_0, \phi(x_0))$ is called a supporting line at x_0 if it always lies below the graph of ϕ i.e. if

$$\phi(x) \geq m(x - x_0) + \phi(x_0)$$

Also we know that if $x, y, x', y' \in (a,b)$ s.t. $x \leq x' < y'$ and $x < y \leq y'$ then

$$\frac{\phi(y) - \phi(x)}{y - x} \leq \frac{\phi(y') - \phi(x')}{y' - x'}$$

$\Rightarrow y = m(x - x_0) + \phi(x_0)$ is supporting line if

$$m \leq \frac{\phi(x) - \phi(x_0)}{x - x_0}$$

i.e. if m lies between left and right hand derivatives at x_0.

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$\because$ if $x \to x_0^+$ then

$$m \le \frac{\phi(x_0+h) - \phi(x_0)}{h}$$

and if $x \to x_0^-$

$$\Rightarrow m \le \frac{\phi(x_0-h) - \phi(x_0)}{-h}$$

$$\Rightarrow m \geq \frac{\phi(x_0-h) - \phi(x_0)}{h}$$

So supporting line always exists at each pt.

<u>Proposition 20</u> (Jensen Inequality)

let ϕ be a convex function on $(-\infty, \infty)$ and f an integrable function on $[0,1]$. Then

$$\int_0^1 \phi(f(t))\,dt \geq \phi\left[\int_0^1 f(t)\,dt\right]$$

<u>Proof</u> let $\alpha = \int_0^1 f(t)\,dt$

and let $y = m(x-\alpha) + \phi(\alpha)$ be equation of supporting line at α.

So $\phi(f(t)) \geq m(f(t)-\alpha) + \phi(\alpha)$

$$\Rightarrow \int_0^1 \phi(f(t))\,dt \geq m\int_0^1 f(t)\,dt - \alpha m\int_0^1 dt + \int_0^1 \phi(\alpha)\,dt$$

$$\Rightarrow \int_0^1 \phi(f(t))\,dt \geq m\alpha - m\alpha + \cancel{\int \phi(f(t))\,dt} \int_0^1 \phi(\alpha$$

$$\Rightarrow \int_0^1 \phi(f(t)\,dt) \geq \phi\left[\int_0^1 f(t)\,dt\right]$$

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Corollary let f be an integrable function on $[0,1]$
Then

$$\int \exp(f(t))\,dt \geq \exp\left[\int f(t)\,dt\right]$$

Proof let $\phi(x) = e^{x}$ then ϕ is convex on $(-\infty,\infty)$ and so by Jensen inequality

$$\int \exp(f(t))\,dt \geq \exp \int f(t)\,dt.$$

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Chapter-4

Def. Let N_1 be a linear space (or vector space) over the field $F (= \mathbb{R}$ or $\mathbb{C})$. A real valued function $f : N_1 \to \mathbb{R}$ is said to define a norm on N_1 if $\forall\, x, y \in N_1$ and for every $\alpha \in F$, f satisfies the following conditions:

(1) $f(x) \geq 0$

(2) $f(x) = 0 \iff x = 0$

(3) $f(\alpha x) = |\alpha|\, f(x)$

(4) $f(x+y) \leq f(x) + f(y)$

We denote $f(x)$ by $\|x\|$

A linear space N_1 together with a norm defined on it is called a <u>Normed Linear Space</u>.

Def. A normed space is said to be <u>complete</u> if every Cauchy sequence in it is convergent. [i.e. for every Cauchy seq. $\{x_n\}$ in N_1, $\exists$ an element x in N_1 s.t. $x_n \to x$].

Def. A complete normed linear space is called <u>Banach Space</u>.

Def. If p and q are +ve real numbers such that $p+q = pq$ or $\frac{1}{p} + \frac{1}{q} = 1$, then we call p and q, a pair of <u>conjugate exponents</u>.

Def. Let $p \in \mathbb{R}$, $p > 0$. We define $L^p = L^p[0,1]$ to be set of all real-valued functions on $[0,1]$ such that

(1) f is measurable and

(2) $\|f\|_p = \left(\int_0^1 |f|^p\right)^{1/p} < \infty$.

[Let $f, g \in L^p$

then $f+g$ is measurable and

$$\|f+g\|_p = \left(\int_0^1 |f+g|^p\right)^{1/p} \leq \left(\int_0^1 (|f|+|g|)^p\right)^{1/p}$$

$$\leq \left(\int_0^1 |f|^p + |g|^p\right)^{1/p}$$

X.

$$\Rightarrow \|f+g\|_p^p \leq \|f\|_p^p + \|g\|_p^p \;]$$

Def. We denote by L^∞, the set of all bounded measurable fns on $[0,1]$.

Here we define

$$\|f\| = \|f\|_\infty = \text{ess sup} |f(t)|,$$

where

$$\text{ess sub } f(t) = \inf \{M ;\ m\{t ; |f(t)| > M\} = 0\}.$$

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Minkowski's Holder Inequalities

Lemma Let α and β be non-neg real no. and $0 < \lambda < 1$. Then

$$\alpha^{\lambda}\beta^{1-\lambda} \le \lambda\alpha + (1-\lambda)\beta$$ with equality iff $\alpha = \beta$.

Proof. Consider the function ϕ defined for non-neg. real no. t by

$$\phi(t) = (1-\lambda) + \lambda t - t^{\lambda}$$

then $\phi'(t) = \lambda(1 - t^{\lambda-1})$.

as $\lambda - 1 < 0 \Rightarrow \phi'(t) < 0$ for $t < 1$ and

$\phi'(t) > 0$ for $t > 1$

So ~~for~~ $t \ne 1$, we have $\phi(t) < \phi(1) = 0$.

By mean value thm, we have that

$$\frac{\phi(t) - \phi(1)}{t-1} \lambda 0$$

$\exists$ c, c_1 s.t.

$$0 > \phi'(c) = \frac{\phi(1) - \phi(t)}{1-t} \quad [\text{if } t < 1] \quad \text{—(1)}$$

$$\text{and } 0 < \phi'(c_1) = \frac{\phi(t) - \phi(1)}{t-1} \quad [\text{if } t > 1] \quad \text{—(2)}$$

from (1) and (2)

$$\phi(1) - \phi(t) < 0 \quad \Rightarrow \quad \phi(t) > \phi(1) = 0.$$

and $\phi(t) = \phi(1) \Leftrightarrow t = 1$

$\Rightarrow$ $\phi(t) \geq 0$ and $\phi(t) = 0$ iff $t = 1$.

Put $t = \alpha/\beta$ if $\beta \neq 0$.

then $$(1-\lambda) + \frac{\lambda\alpha}{\beta} - \left(\frac{\alpha}{\beta}\right)^{\lambda} \geq 0$$ with equality iff $\alpha = \beta$.

i.e. $(1-d)\beta + d\alpha \geq \alpha^{d}\beta^{1-d}$

with equality iff $\alpha = \beta$.

<u>Holder Inequality</u> If p and q are non-ne[gative] extended real no. such that

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \text{and if } f \in L^p, \; g \in L^q \text{ th[en]}$$

$f \cdot g \in L^1$ and $\int_0^1 fg \leq \|f\|_p \cdot \|g\|_q$.

Equality holds iff for some non-zero constants α and β, we have

$$\alpha |f|^p = \beta |g|^q \quad a.e.$$

<u>Proof:-</u> If $p = 1$ then $q = \infty$.

$\Rightarrow$ $f \in L$ and $g \in L^{\infty}$

and $|g| \leq \|g\|_\infty$ a.e.

$\Rightarrow |f| \leq \|f\|_\infty$ a.e.

$\Rightarrow |f+g| \leq |f| + |g| \leq \|f\|_\infty + \|g\|_\infty$ a.e.

$\Rightarrow \|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$.

then by def. of $\|g\|_\infty$,

$$|g| \leq \|g\|_\infty \quad a.e. \text{ in } [0,1]$$

$\Rightarrow |fg| = |f||g| \leq \|g\|_\infty |f| \quad a.e. \text{ in } [0,1]$

$\Rightarrow fg \in L^1$ &

$$\int_0^1 |fg| \leq \|g\|_\infty \int_0^1 |f| \leq \|f\|_1 \|g\|_\infty.$$

So we assume that $1 < p < \infty$ and so $1 < q < \infty$.

Firstly suppose that $\|f\|_p = \|g\|_q = 1$ and apply Lemma 1 with $\alpha = |f(t)|^p$, $\beta = |g(t)|^q$

$\lambda = \frac{1}{p}$, $1-\lambda = \frac{1}{q}$. Then

$$|f(t)g(t)| \le \lambda |f(t)|^p + (1-\lambda)|g(t)|^q \quad \text{--- (1)}$$

Integrating both sides, we get

$$\int |fg| \le \lambda \int |f|^p + (1-\lambda) \int |g|^p = 1 \quad \text{--- (2)}$$

Also if $\|f\| = 0$ or $\|g\| = 0$, then inequality is trivial.

So $f \in L^p$ and $g \in L^q$ s.t. $\|f\| \neq 0$ and $\|g\| \neq 0$.

then $\left\| \frac{f}{\|f\|_p} \right\| = \left\| \frac{g}{\|g\|_q} \right\| = 1$.

So by above discussion

$$\int \left| \frac{f}{\|f\|_p} \cdot \frac{g}{\|g\|_q} \right| \le 1$$

$$\Rightarrow \int |fg| \le \|f\|_p \|g\|_q$$

$$\Rightarrow \int_0^1 fg \le \|f\|_p \|g\|_q$$

Also equality in (1) occurs iff

$$|f(t)|^p = |g(t)|^q$$

and equality in (2) occurs iff it holds a.e. in (1)

$\Rightarrow$ equality holds iff

$$\cancel{|f(t)g(t)|} \left| \frac{f}{\|f\|} \right|^p = \left| \frac{g}{\|g\|} \right|^q \quad \text{a.e.}$$

i.e. $\|g\|^q |f|^p = \|f\|^p |g|^q$ a.e

Minkowski Inequality

If f and g are in L^p, then so is $f+g$

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

Proof. If $p=1$, the result is obvious.

If $p=\infty$ then

$$|f| \leq \|f\|_\infty \text{ a.e. } \quad \& \quad |g| \leq \|g\|_\infty \text{ a.e.}$$

$$\Rightarrow |f+g| \leq |f|+|g| \leq \|f\|_\infty + \|g\|_\infty \text{ a.e.}$$

$$\Rightarrow \|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty.$$

So we can assume that $1 < p < \infty$.

Since

$$|f+g|^p \leq |f|^p + |g|^p$$

$$\Rightarrow f+g \in L^p$$

Also $$\int |f+g|^p \leq \int |f+g|^{p-1}|f| + \int |f+g|^{p-1}|g|$$

By Holder Inequality,

$$\int |f+g|^{p-1}|f| \leq \|f\|_p \|(|f+g|^{p-1})\|_q,$$

$$\int |f+g|^{p-1}|g| \leq \|g\|_p \|(|f+g|^{p-1})\|_q$$

and

$$\|(|f+g|^{p-1})\|_q = \left\{ \int |f+g|^{(p-1)q} \right\}^{1/q}$$

$$= \left\{ \|f+g\|_p \right\}^{p/q} \quad [\because q(p-1)=p]$$

$$\Rightarrow \|f+g\|_p^p \leq (\|f\|_p + \|g\|_p)(\|f+g\|_p)^{p/q}$$

$$\Rightarrow \|f+g\|_p \leq \|f\|_p + \|g\|_p$$

$P+Q$ Def. A seq. $\langle f_n \rangle$ in a normed linear space is said to converge to an element f in the space if given $\epsilon > 0$, $\exists$ N s.t. for all $n > N$, we have

$$\|f - f_n\| < \epsilon .$$

If f_n converges to f, we write $f = \lim f_n$ or $f_n \to f$.

Convergence in the mean of order P – A seq. of fns $\langle f_n \rangle$ is said to converge to f in the mean of order ~~p~~ P if each $f_n \in L^P$ and

$$\|f - f_n\|_p \to 0 .$$

Cauchy Seq. – A seq. $\langle f_n \rangle$ in a normed linear space is a Cauchy seq. if, given $\epsilon > 0$, $\exists$ N s.t. $\forall$ $n \geq N$ and $\forall$ $m \geq N$, we have

$$\|f_n - f_m\| < \epsilon .$$

Def. A normed linear space is called Complete if every Cauchy sequence in the space converges, i.e., for each Cauchy seq. $\langle f_n \rangle$ in the space, $\exists$ an element f in the space such that $f_n \to f$.

A Complete normed linear space is called Banach Space.

Summable series – A series $\langle f_n \rangle$ in a normed linear space is said to be summable to a sum s if s is in the space and the seq.

of partial sums of the series converges to

i.e

$$\|s - \sum_{i=1}^{n} f_i\| \to 0.$$

Notation we write $s = \sum_{i=1}^{\infty} f_i$.

Absolutely Summable Series The series $\langle f_n \rangle$ is said to be absolutely summable if

$$\sum_{n=1}^{\infty} \|f_n\| < \infty.$$

Note - For a series of real numbers, absolute summability implies that the series is summable.

But this is not true in general for series of elements in a normed linear space.

Prop: A normed linear space is complete iff every absolutely summable series is summable.

Proof let X be complete normed linear space and $\langle f_n \rangle$ be absolutely summable series of elements of X.

Then $\sum_{n=1}^{\infty} \|f_n\| < \infty$

$\Rightarrow$ given $\epsilon > 0$, $\exists$ ~~no~~ N_ϵ s.t.

$$\sum_{n=N}^{\infty} \|f_n\| < \epsilon.$$

Now let $S_n = \sum_{i=1}^{n} f_i$ be the partial sum of the series $\langle f_n \rangle$

Then for $n \geq m \geq N$, we have

$$\|S_n - S_m\| = \left\|\sum_{i=1}^{n} f_i - \sum_{i=1}^{m} f_i\right\| = \left\|\sum_{i=m}^{n} f_i\right\|$$

$$\leq \sum_{i=m}^{n} \|f_i\| \leq \sum_{i=N}^{\infty} \|f_i\| < \epsilon.$$

Hence the sequence $\langle S_n \rangle$ of partial sums is a Cauchy sequence and must converge to an element s in X. ($\because X$ is complete)

<u>Converse</u> - Suppose that every absolute summable series in X is summable.

Let $\langle f_n \rangle$ be a Cauchy seq. in X.

$\Rightarrow$ Given $0 < \epsilon = 2^{-k}$ (say) [i.e. for given integer k]

$\exists$ an integer n_k s.t.

$\|f_n - f_m\| < 2^{-k}$ for all $n, m \geq n_k$.

Also we can choose n_k's so that $n_{k+1} > n_k$.

Then $\langle f_{n_k} \rangle_{k=1}^{\infty}$ is a subsequence of $\langle f_n \rangle$

and setting $g_1 = f_{n_1}$, $g_k = f_{n_k} - f_{n_{k-1}}$ for $k > 1$,

we obtain a series $\langle g_k \rangle$ whose k^{th} partial sum is f_{n_k}.

But $\|g_k\| = \|f_{n_k} - f_{n_{k-1}}\| \leq 2^{-(k-1)} = 2^{-k+1}$ for $k > 1$

$\Rightarrow \sum \|g_k\| \leq \|g_1\| + \sum 2^{-k+1} = \|g_1\| + 1$

$\Rightarrow$ series $\langle g_k \rangle$ is absolutely summable and hence summable.

$\Rightarrow$ Partial sums of the series converge

$\Rightarrow$ $\langle f_{n_k} \rangle \to f$.

Claim $f = \lim f_n$.

As $\langle f_n \rangle$ is a Cauchy sequence, given $\epsilon > 0$,

$\exists$ N s.t. $\|f_n - f_m\| < \epsilon/2$ $\forall$ $n, m > N$.

as $f_{n_k} \to f$, $\exists$ k' s.t. $\forall$ $k \geq k'$, we have

$\|f_{n_k} - f\| < \epsilon/2$.

Take k to be so large that $k > k'$ and

$n_k \geq N$. Then

$\|f_n - f\| \leq \|f_n - f_{n_k}\| + \|f_{n_k} - f\| \leq \epsilon/2 + \epsilon/2 = \epsilon$

$\Rightarrow$ $\forall$ $n > N$

$\|f_n - f\| < \epsilon$ $\Rightarrow$ $f_n \to f$.

Riesz-Fischer Theorem - L^p spaces are complete.

Proof. The case $p = \infty$ is elementary.

Assume $1 \leq p < \infty$.

we only need to show that every absolutely summable series in L^p is summable in L^p to some element of L^p.

let $\langle f_n \rangle$ be series in L^p with $\sum_{n=1}^{\infty} \|f_n\| = M < \infty$.

Define functions g_n by setting

$$g_n(x) = \sum_{k=1}^{n} |f_k(x)|$$

From Minkowski Inequality, we have

$$\|g_n\| \leq \sum_{k=1}^{n} \|f_k\| \leq M.$$

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Hence $\int (g_n)^p \leq M^p$.

for each x, $\langle g_n(x) \rangle$ is an increasing seq. of (extended) real numbers and so must converge to an extended real number $g(x)$.

Also the function g is measurable [$\because$ each f_i is measurable]. and as $g_n \geq 0$, we have (by Fatou's lemma)

$$\int g^p \leq \lim_{n\to\infty} \int (g_n)^p \leq M^p$$

$$\Rightarrow \int g^p \leq M^p$$

$\Rightarrow$ g^p is integrable and so $g(x)$ is finite for almost all x.

for each x, such that $g(x)$ is finite, the series $\sum_{k=1}^{\infty} f_k(x)$ is an absolutely summable series of real numbers and so must be summable to a real number $s(x)$.

Setting $s(x) = 0$ for those x where $g(x) = \infty$, we get a fn. s, which is the limit a.e. of the partial sums

$$s_n = \sum_{k=1}^{n} f_k.$$

$\Rightarrow$ s is measurable.

Also $|s_n(x)| \leq g(x) \Rightarrow |s(x)| \leq g(x)$

$$\Rightarrow \int |s(x)|^p \leq \int g^p \leq M^p$$

$$\Rightarrow s \in L^p.$$

$\Rightarrow$ Partial sums of ...

and $|S_n(x) - S(x)|^P \le (|S_n(x)| + |S(x)|)^P$

$$\le (g(x) + g(x))^P$$

$$= 2^P (g(x))^P.$$

as $2^P g^P$ is integrable and

$|S_n(x) - S(x)|^P \to 0$ for almost all x

$$\Rightarrow \int |S_n - S|^P \to 0$$

(by Lebesgue convergence thm.)

$\Rightarrow \|S_n - S\|^P \to 0$

$\Rightarrow \|S_n - S\| \to 0$

Hence the series $\langle f_n \rangle$ has sum $S \in L^P$.

Bounded Linear functionals on L^P spaces

Def. A linear functional on a normed linear space X is a mapping $F : X \to \mathbb{R}$ s.t.

$$F(\alpha f + \beta g) = \alpha F(f) + \beta F(g).$$

We say that the linear functional is bounded if $\exists$ a constant M s.t. $|F(f)| \le M . \|f\| \ \forall f \in X$.

Norm of F $\quad \|F\| = \sup \frac{|F(f)|}{\|f\|}$ as f ranges over all non-zero elements of X.